

ON J -HOLOMORPHIC CURVES IN ALMOST COMPLEX MANIFOLDS WITH ASYMPTOTICALLY CYLINDRICAL ENDS

ERKAO BAO

Abstract

The compactification of moduli spaces of J -holomorphic curves in almost complex manifolds with cylindrical ends is crucial in Symplectic Field Theory. One natural generalization is to replace “cylindrical” by “asymptotically cylindrical”. In this article we generalize the compactness results from [14, 15, 4, 5] to this setting. As an application, we study the relation between the moduli spaces of J -holomorphic polygons before and after the Lagrangian surgery established in [11] in a more general setting and from a different viewpoint.

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1 Introduction

Introduced by Gromov in 1985, J -holomorphic curves have been studied intensively in closed symplectic manifolds. In 1993 Hofer studied the behaviors of J -holomorphic curves in symplectization of contact manifolds, which is non-compact. Shortly after that, Eliashberg, Givental and Hofer invented the Symplectic Field Theory, which greatly helps us understand symplectic manifolds and contact manifolds. In most of previous literature, the only noncompact almost complex manifolds studied are the ones with cylindrical ends, which roughly means that the almost complex structure J is translationally invariant near the ends. In [5] the notion of asymptotically cylindrical almost complex structure was introduced, which is a natural generalization of cylindrical almost complex structure. However, there is no corresponding result proven for asymptotically cylindrical almost complex structure. Intuitively, we should have similar results as in the cylindrical case. However, the original proofs rely heavily on the cylindrical nature of the almost complex structure, which prevents us from applying them directly to the asymptotically cylindrical case. In this paper, we give the modified definition of asymptotically cylindrical almost complex structure, and prove some parallel analytical results as in cylindrical case. Based on these results we can compactify the moduli space of J -holomorphic curves in almost complex manifolds with asymptotically cylindrical ends using the idea of holomorphic buildings introduced by [5].

One of the advantages of this generalization lies in applications. In many cases the original almost complex structure is asymptotically cylindrical, so if we insist on getting a cylindrical almost complex structure we have to perturb the original one in a special way, which may turn a generic almost complex structure into a non-generic one. However, using the results of this paper, we can work directly in the asymptotically cylindrical context. We also take this chance to fill in the gaps between different literatures.

In the asymptotically cylindrical case, the proofs of some theorems are significantly different and more difficult than the proofs in the cylindrical case, for example, Proposition 2, Theorem 1 and Theorem 3. The difficulties mainly come from the following two facts: 1. the obvious translations in the cylindrical almost complex manifold are not J -holomorphic anymore; 2. Hofer's energy cannot be chosen to be both non-negative and exact.

In Section 2, we give the definition of asymptotically cylindrical almost complex manifolds and the definition of Hofer's energy of J -holomorphic curves in this context. To the best of the author's knowledge, the sort of conditions of

(AC3) and (AC4) in Definition 1 have not appeared in the literatures in Symplectic Geometry, but similar conditions appear very often in global Riemannian Geometry. We also list the main analytical results of this paper, not including the application.

In Section 3, we give the proofs of the main results listed in Section 2. The proofs follow the schemes of [14, 15, 16, 4, 5]. However, some proofs are quite different and more difficult.

In Section 4, we give the definition of almost complex manifolds with asymptotically cylindrical ends and the definition of Hofer's energy of J -holomorphic curves in this context.

In Section 5, we apply the results developed in the previous sections to Lagrangian surgery. In particular, we give a simpler and more natural proof for Theorem Z in [11] in a more general context. Theorem Z is the main theorem in [11] with a lot of applications. It is a fundamental theorem to get long exact sequences of Lagrangian Floer homology, whose relations to Mirror Symmetry can be found in [8, 11]. We could also use Theorem Z to understand wall crossing formula (see [11]). In this paper, we compactify the moduli space arises in Theorem Z, carry out the scale dependent gluing (see [20]) of certain moduli spaces, and prove the surjectivity of the gluing map to complete the proof of Theorem Z. All these steps are done in the spirit of Symplectic Field Theory rather than purely following a bunch of estimates as in [11]. Also, in [11] Theorem Z is proved under the assumption that the almost complex structure is integrable in a neighborhood of the intersection point of two Lagrangian submanifolds. In this paper, we do not need the integrability assumption. In suitable polar coordinates around the intersection point, an integrable almost complex structure corresponds to a cylindrical almost complex structure, while an almost complex structure corresponds to an asymptotically cylindrical almost complex structure. There will be more applications to appear in the second paper.

In the appendix, we include the basic notions of the moduli space of bordered stable nodal Riemann surfaces and the stable map topology of holomorphic buildings.

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2 Asymptotically cylindrical almost complex manifolds

2.1 Definition

Let V be a smooth closed manifold of dimension $2n + 1$, J be a smooth almost complex structure in $W = \mathbb{R} \times V$, $\mathbf{R} := J(\frac{\partial}{\partial r})$ be a smooth vector field on W , and ξ be a subbundle of the tangent bundle TW of W defined by $\xi(r, v) = (JT_v\{r\} \times V) \cap (T_v\{r\} \times V)$, for $(r, v) \in W$. Then tangent bundle TW splits as $TW = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$. Define a 1-form λ on W by: $\lambda(\xi) = 0$, $\lambda(\frac{\partial}{\partial r}) = 0$, $\lambda(\mathbf{R}) = 1$, and a 1-form σ on W by: $\sigma(\xi) = 0$, $\sigma(\frac{\partial}{\partial r}) = 1$, $\sigma(\mathbf{R}) = 0$.

Let $f_s : W \rightarrow W$ be the translation $f_s(r, v) = (r + s, v)$. We call a tensor on W translationally invariant if it is invariant under f_s .

Definition 1. Under the above notations, J is called asymptotically cylindrical at positive infinity, if there exists a 2-form ω on W such that for some integer $l \geq 4$ the pair (J, ω) satisfies (AC1)-(AC7):

- (AC1) $i(\frac{\partial}{\partial r})\omega = 0 = i(\mathbf{R})\omega$.
- (AC2) $\omega|_{\xi}(\cdot, J\cdot)$ is a metric on ξ .¹
- (AC3) There exists a smooth translationally invariant almost complex structure J_∞ on W and constants $C_l, \delta_l \geq 0$, such that

$$\left\| (J - J_\infty)|_{[r, +\infty) \times V} \right\|_{C^l} \leq C_l e^{-\delta_l r} \quad (1)$$

for all $r \geq 0$, where $\|\varphi\|_{C^l} := \sup_w \sum_{k=0}^l |\nabla^k \varphi(w)|$ and $|\cdot|$ is computed using a translationally invariant metric g_W on W , for example $g_W = dr^2 + g_V$, and ∇ is the corresponding Levi-Civita connection.²

- (AC4) There exists a translationally invariant closed 2-form ω_∞ on W and constants $C_l, \delta_l \geq 0$, such that

$$\left\| (\omega - \omega_\infty)|_{[r, +\infty) \times V} \right\|_{C^l} \leq C_l e^{-\delta_l r} \quad (2)$$

for all $r \geq 0$.

- (AC5) The pair $(\omega_\infty, J_\infty)$ satisfies (AC1) and (AC2).
- (AC6) $i(\mathbf{R}_\infty)d\lambda_\infty = 0$, where $\mathbf{R}_\infty := \lim_{s \rightarrow \infty} f_s^* \mathbf{R}$, $\lambda_\infty := \lim_{s \rightarrow \infty} f_s^* \lambda$, and both limits exist by (AC3).

¹ Actually we only need $\omega|_{\xi}(u, Ju) > 0$ for all $0 \neq u \in \xi$, but for convenience we also require $\omega|_{\xi}(Ju, Jv) = \omega|_{\xi}(u, v)$.

² Equivalently, we could embed V into \mathbb{R}^N and take the derivatives there, and $\|\varphi\|_{C^l} = \sup_w \sum_{|\alpha|=0}^l |D^\alpha \varphi(w)|$.

- (AC7) $\mathbf{R}_\infty(r, v) = J_\infty\left(\frac{\partial}{\partial r}\right) \in T_v(\{r\} \times V)$.

Remark 1. The terminology we use is slightly different from the one in [5]. One major difference is that we require J converges to J_∞ exponentially fast in condition (AC3). It seems that we should call J “exponentially asymptotically cylindrical almost complex structure”, but since this is the only case we consider in our paper, we omit the word “exponentially”. Also this is the accurate condition to guarantee that the J -holomorphic curve converges to the periodic orbits of \mathbf{R}_∞ exponentially fast by the footnote of formula (40).

Similarly, we could define the notion of J being asymptotically cylindrical at negative infinity. If J is asymptotically cylindrical at both positive infinity and negative infinity, we say J is asymptotically cylindrical. When we say J is asymptotically cylindrical, we choose ω without mentioning.

The following definition is the case considered in [14, 15, 16, 4, 5].

Definition 2. We say J is a cylindrical almost complex structure, if J is an asymptotically cylindrical almost complex structure and both J and ω are all translationally invariant.

Example 1. (Symplectization) Assume (V, ξ) is a contact manifold with contact 1-form λ and Reeb vector field \mathbf{R} , i.e. $\xi = \ker \lambda$, $\lambda \wedge (d\lambda)^n \neq 0$, $i_{\mathbf{R}}d\lambda = 0$, and $\lambda(\mathbf{R}) = 1$. Let $\omega = d\lambda$ and let J_ξ be an almost complex structure in ξ such that it is compatible with $\omega|_\xi$, i.e. $d\lambda(\cdot, J_\xi \cdot)$ is a metric on ξ . We extend the J_ξ to the tangent bundle of $W = \mathbb{R} \times V$ by setting $J(\frac{\partial}{\partial r}) = \mathbf{R}$. Then (W, J) is a cylindrical almost complex manifold and in particular an asymptotically cylindrical almost complex structure.

Refer to [5] for other interesting examples of cylindrical almost complex manifolds.

Example 2. Assume J is a smooth almost complex structure on \mathbb{R}^{2n+2} with $J(0) = J_0(0)$, where J_0 is the standard complex structure on \mathbb{R}^{2n+2} . Assume ω' is a smooth symplectic form on \mathbb{R}^{2n+2} with $\omega'(0) = \sum_{i=1}^{n+1} dx_i \wedge dy_i$. Assume that J is ω' compatible, i.e. $\omega'(\cdot, J\cdot)$ is a Riemannian metric on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Consider $\mathbb{R}^{2n+2} \setminus \{0\}$ and pick an polar coordinate chart, i.e.

$$\begin{aligned} \varphi : \mathbb{R}^- \times S^{2n+1} &\rightarrow \mathbb{R}^{2n+2} \setminus \{0\}, \\ (r, \Theta) &\mapsto e^r \Theta, \end{aligned}$$

where we view S^{2n+1} as the unit sphere inside \mathbb{R}^{2n+2} .

We can define ξ and \mathbf{R} as in the beginning of 2.1. Denote the projection: $T(\mathbb{R}^- \times S^{2n+1}) = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi \rightarrow \xi$ by π_ξ . Define a 2-form ω on $\mathbb{R}^- \times S^{2n+1}$ by $\omega(u, v) = e^{-2r} \omega'(\varphi_* \pi_\xi u, \varphi_* \pi_\xi v)$, where $u, v \in T_{(r, \theta)} \mathbb{R}^- \times S^{2n+1}$. Then $\omega_{-\infty} = \Pi^* \omega_0$, where $\Pi : \mathbb{R}^- \times S^{2n+1} \rightarrow S^{2n+1}$ is the projection, $\omega_0 := i^* (\sum_{i=1}^{n+1} dx_i \wedge dy_i)$, and $i : S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$ is the embedding.

Now it is clear that $(\mathbb{R}^- \times S^{2n+1}, J)$ is an asymptotically cylindrical almost complex manifold near $-\infty$.

By (AC3) and (AC7) we can see that \mathbf{R}_∞ is a translationally invariant vector field on W and it is tangent to each level set $\{r\} \times V$, so we can view \mathbf{R}_∞ as a vector field on V . Let ϕ^t be the flow of \mathbf{R}_∞ on V , i.e. $\phi^t : V \rightarrow V$ satisfies $\frac{d}{dt}\phi^t = \mathbf{R}_\infty \circ \phi^t$. Then we have

$$\frac{d}{dt}[(\phi^t)^*\lambda_\infty] = (\phi^t)^*(i_{\mathbf{R}_\infty}d\lambda_\infty + di_{\mathbf{R}_\infty}\lambda_\infty) = 0.$$

Thus ϕ^t preserves λ_∞ so ξ_∞ . Similarly ϕ^t preserves ω_∞ .

Let's denote by \mathcal{P} the set of periodic trajectories, counting their multiples, of the vector field \mathbf{R}_∞ restricting to V . Notice that any smooth family of periodic trajectories from \mathcal{P} has the same period by Stoke's Theorem.

Definition 3. A T -periodic orbit γ of \mathbf{R}_∞ is called non-degenerate, if $d\phi^T|_{\xi(\gamma(0))}$ does not have 1 as an eigenvalue, where ϕ^t is the flow of \mathbf{R}_∞ . We say that J is non-degenerate if all the periodic solutions of \mathbf{R}_∞ are non-degenerate.

A weaker requirement for J than non-degenerate is Morse-Bott.

Definition 4. We say that J is of the Morse-Bott type if, for every $T > 0$ the subset $N_T \subset V$ formed by the closed trajectories from \mathcal{P} of period T is a smooth closed submanifold of V , such that the rank of $\omega_\infty|_{N_T}$ is locally constant and $T_p N_T = \ker(d\phi^T - Id)_p$.

We always assume J is of Morse-Bott type in this paper.

2.2 Energy of J -holomorphic curves

Let Σ be a punctured Riemann surface with almost complex structure j , and $\tilde{u} = (a, u) : (\Sigma, j) \rightarrow (W, J)$ be a J -holomorphic curve, i.e. $T\tilde{u} \circ j = J(\tilde{u}) \circ T\tilde{u}$. The following definition is a modification of Hofer's energy in cylindrical almost complex structure case. The ω -energy and λ -energy are defined as follows respectively

$$E_\omega(\tilde{u}) = \int_{\Sigma} \tilde{u}^* \omega,$$

$$E_\lambda(\tilde{u}) = \sup_{\phi \in \mathcal{C}} \int_{\Sigma} \tilde{u}^*(\phi(r)\sigma \wedge \lambda),$$

where $\mathcal{C} = \{\phi \in C_c^\infty(\mathbb{R}, [0, 1]) \mid \int_{-\infty}^{+\infty} \phi(x)dx = 1\}$ ³, and λ, σ are defined as in the beginning of subsection 2.1. Let's define the energy of \tilde{u} by

$$E(\tilde{u}) = E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

Equip $\mathbb{R}^+ \times S^1$ with the standard complex structure and coordinate (s, t) , and consider a J -holomorphic map $\tilde{u} = (a, u) : \mathbb{R}^+ \times S^1 \rightarrow W = \mathbb{R} \times V$, Let's

³In [5], the set \mathcal{C} is given by $\mathcal{C} = \{\phi \in C_c^\infty(\mathbb{R}, \mathbb{R}^+) \mid \int_{-\infty}^{+\infty} \phi(x)dx = 1\}$. It is easier to get uniform energy bounds using the modified definition in the case when the almost complex structure is only asymptotically cylindrical.

denote the projections from $TW = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$ to each subbundle by $\pi_r, \pi_{\mathbf{R}}$ and π_ξ . Here we view S^1 as \mathbb{R}/\mathbb{Z} . Notice

$$\begin{aligned}\tilde{u}^*\omega &= \omega(\tilde{u}_s, \tilde{u}_t)ds \wedge dt \\ &= \omega(\tilde{u}_s, J(\tilde{u})\tilde{u}_s)ds \wedge dt \\ &= \omega(\pi_\xi \tilde{u}_s, J(\tilde{u})\pi_\xi \tilde{u}_s)ds \wedge dt\end{aligned}\tag{3}$$

and

$$\begin{aligned}\tilde{u}^*(\phi(r)\sigma \wedge \lambda) &= \phi(a) [\sigma(\tilde{u}_s)\lambda(J(\tilde{u})\tilde{u}_s) - \sigma(J(\tilde{u})\tilde{u}_s)\lambda(\tilde{u}_s)] ds \wedge dt \\ &= \phi(a) [\sigma(\tilde{u}_s)^2 + \lambda(\tilde{u}_s)^2] ds \wedge dt.\end{aligned}\tag{4}$$

Thus, we have $E_\omega(\tilde{u}) \geq 0$ and $E_\lambda(\tilde{u}) \geq 0$.

2.3 Main Results

The following two theorems tell us the behaviors of J -holomorphic curves near infinity.

Theorem 1. *Suppose that $(W = \mathbb{R} \times V, J)$ is an asymptotically cylindrical almost complex manifold. Let $\tilde{u} = (a, u) : \mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \rightarrow W$ be a finite energy J -holomorphic curve. Suppose that the image of \tilde{u} is unbounded in $\mathbb{R} \times V$. Then there exists a periodic orbit γ of \mathbf{R}_∞ of period $|T|$ with $T \neq 0$, such that*

$$\begin{aligned}\lim_{s \rightarrow \infty} u(s, t) &= \gamma(Tt) \\ \lim_{s \rightarrow \infty} \frac{a(s, t)}{s} &= T\end{aligned}$$

in $C^\infty(S^1)$.

It is much more difficult to prove this theorem in the asymptotically cylindrical case compared to the cylindrical case.

The above theorem tells us that when s is large enough $u(s, t)$ lies inside a small neighborhood of γ . We will construct a coordinate chart for such neighborhood $U \subset S^1 \times \mathbb{R}^{2n} \rightarrow V$, and then we can view the map \tilde{u} as

$$\tilde{u}(s, t) = (a(s, t), \vartheta(s, t), z(s, t)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n},$$

where ϑ is the coordinate of the universal cover of S^1 .

Theorem 2. *Under the same assumption as in Theorem 1, there exist constants $M_\beta, d_\beta, a_0, \vartheta_0, s_0 > 0$ such that*

$$\begin{aligned}|D^\beta \{a(s, t) - Ts - a_0\}| &\leq M_\beta e^{-d_\beta s}, \\ |D^\beta \{\vartheta(s, t) - Tt - \vartheta_0\}| &\leq M_\beta e^{-d_\beta s}, \\ |D^\beta \{z(s, t)\}| &\leq M_\beta e^{-d_\beta s},\end{aligned}$$

for all $s > s_0$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N} \times \mathbb{N}$ with $|\beta| = \beta_1 + \beta_2 \leq l - 2$, where l is the integer in Definition 1.

3 Proof of main results

The proof is done in three steps. The first step is to show that the gradient of a finite energy J -holomorphic curve $\tilde{u} = (a, u)$ is bounded. The second step is to show subsequence convergence, briefly, given a sequence of numbers R_k converging to infinity, we want to show that there exists a subsequence R_{k_n} , such that $u(R_{k_n}, t)$ converges to a periodic solution of the vector field \mathbf{R}_∞ . The third step is to get some exponential estimate and then prove the Theorem 1 and Theorem 2.

3.1 Gradient bounds

We cite the following two lemmata for later use.

Lemma 1. [14] *Let (X, d) be a metric space. Equivalent are*

- (1) *(X, d) is complete.*
- (2) *For every continuous map $\phi : X \rightarrow [0, +\infty)$ and given $x \in X$, $\varepsilon > 0$ there exist $x' \in X$, $\varepsilon' > 0$ such that*

- $\varepsilon' \leq \varepsilon$, $\phi(x')\varepsilon' \geq \phi(x)\varepsilon$,
- $d(x, x') \leq 2\varepsilon$,
- $2\phi(x') \geq \phi(y)$ for all $y \in X$ with $d(y, x') \leq \varepsilon'$.

Let J be an asymptotically cylindrical almost complex structure on $W = \mathbb{R} \times V$, let $\tilde{u} = (a, u)$ be a J -holomorphic map from $B(0, R)$ to W , where $B(z_0, R) := \{s + \sqrt{-1}t \in \mathbb{C} \mid |z - z_0| < R\}$, denote

$$\|\nabla \tilde{u}\| := \sup_{(s,t) \in B(0,R)} |\nabla \tilde{u}(s, t)| \quad (5)$$

and

$$\|\tilde{u}\|_{C^k(B(0,R),W)} := \sup_{x \in B(0,R)} \sum_{|l|=0}^k |\nabla^l \tilde{u}(x)|,$$

where the norm $|\cdot|$ is computed with respect to the standard metric $ds^2 + dt^2$ on $B(z_0, R)$ and a translationally invariant metric g_W on W , for example $g_W = g_V + dr^2$, and ∇ is the the Levi-Civita connection with respect to g_W on W . Equivalently, we could embed V into \mathbb{R}^N and take derivatives there and compute the C^k -norm. The following lemma says that the gradient bound implies C^∞ bound.

Lemma 2. (Gromov-Schwarz) Fix $0 < \varepsilon < 1$ and $k \in \mathbb{N}$, if $\|\nabla \tilde{u}\| < C' < +\infty$, then there exists $C(k, C') > 0$ such that

$$\|\tilde{u}\|_{C^k(B(0, R-\varepsilon), W)} \leq C(k, C'),$$

where $C(k, C')$ does not depend on \tilde{u} .

Proof. This is a standard result. Using the gradient bound of \tilde{u} , we can find uniform coordinate charts both in domain and in target, then we can apply Proposition 2.36 in [2]. \square

The following proposition is one of the key steps in [14] whose proof reveals the relation between ω energy and trajectory of \mathbf{R}_∞ .

Proposition 1. [14] Suppose J is a cylindrical almost complex structure on W and $\tilde{u} = (a, u) : \mathbb{C} \rightarrow W$ is a finite energy J -holomorphic plane (i.e. $E(\tilde{u}) = E_\lambda(\tilde{u}) + E_\omega(\tilde{u}) < +\infty$). If $E_\omega(\tilde{u}) = 0$ and $\|\nabla \tilde{u}\| \leq C$ for some $C > 0$, then \tilde{u} is constant.

Proof. Suppose \tilde{u} is not constant. By (3), $\pi_\xi \tilde{u}_s = 0 = \pi_\xi \tilde{u}_t$. Hence $\pi_\xi \circ T\tilde{u}$ is the zero section of $\tilde{u}^* \xi \rightarrow \mathbb{C}$. Therefore we have $u(s, t) = x \circ f(s, t)$, where x is a solution of $\dot{x} = \mathbf{R}(x)$ and $f : \mathbb{C} \rightarrow \mathbb{R}$ is a smooth function. Consequently, $f_s = -a_t$; $f_t = a_s$. Hence $\Phi := f + ia$ is a holomorphic function on \mathbb{C} . Since $\|\nabla \tilde{u}\|$ is bounded, $\|\nabla \Phi\|$ is bounded; thus Φ is a linear function. By (4)

$$\begin{aligned} E_\lambda(\tilde{u}) &= \sup_{\phi \in \mathcal{C}} \int_{\mathbb{C}} \phi(a)(a_s^2 + a_t^2) ds \wedge dt \\ &= c \cdot \sup_{\phi \in \mathcal{C}} \int_{\mathbb{C}} \phi(s) ds \wedge dt \quad (\text{for some } c > 0, \text{ via a linear change of variables}) \\ &= +\infty. \end{aligned}$$

\square

The proposition below generalizes Proposition 27 in [14] to the asymptotically cylindrical case. The proof is much harder.

Proposition 2. If J is an asymptotically cylindrical almost complex structure on W , and \tilde{u} is a J -holomorphic map from \mathbb{C} to W satisfying $E(\tilde{u}) < +\infty$, then we get $\|\nabla \tilde{u}\| < +\infty$.

Proof. Suppose to the contrary, then there exist a sequence of points $z_k \in \mathbb{C}$, satisfying, $|z_k| \rightarrow \infty$, $R_k := \|\nabla \tilde{u}(z_k)\| \rightarrow \infty$, as $k \rightarrow \infty$. By Lemma 1, we can modify z_k such that there exist a sequence of $\varepsilon_k > 0$ satisfying: $\varepsilon_k \rightarrow 0$, $\varepsilon_k R_k \rightarrow +\infty$, and $|\nabla \tilde{u}(z)| \leq 2R_k$ for $z \in B(z_k, \varepsilon_k)$. Now there are two cases.

Case1. $\{a(z_k)\}_{k \in \mathbb{Z}}$ is unbounded.

Then there exist a subsequence of z_k , still denoted by z_k , such that $a(z_k) \rightarrow +\infty$ or $a(z_k) \rightarrow -\infty$. WLOG, let's assume $a(z_k) \rightarrow +\infty$. Pick a further

subsequence of z_k , such that $a(z_k) \geq 2^{k+2}$. Let $\varepsilon'_k := \min \left\{ \varepsilon_k, \frac{2^k}{R_k} \right\}$, then we have $\varepsilon'_k \rightarrow 0$, $\varepsilon'_k R_k \rightarrow +\infty$, and $|a(z) - a(z_k)| \leq 2\varepsilon'_k R_k \leq 2 \cdot \frac{2^k}{R_k} \cdot R_k = 2^{k+1}$, for $|z - z_k| \leq \varepsilon'_k$. Thus, $a(z) \geq a(z_k) - 2^{k+1} \geq 2^{k+2} - 2^{k+1} = 2^{k+1}$, for $|z - z_k| \leq \varepsilon'_k$. Since \tilde{u} is J -holomorphic, we have

$$J(\tilde{u}) \circ T\tilde{u} = T\tilde{u} \circ i. \quad (6)$$

Thus

$$J_\infty(\tilde{u}) \circ T\tilde{u} = T\tilde{u} \circ i + (J_\infty - J)(\tilde{u}) \circ T\tilde{u}. \quad (7)$$

By AC3), we have⁴

$$\delta_k := \sup_{z \in B(z_k, \varepsilon'_k)} \|(J_\infty - J)(\tilde{u}(z))\| \rightarrow 0,$$

as $k \rightarrow +\infty$. Define $l_k : \mathbb{C} \rightarrow \mathbb{C}$ by

$$l_k(z) = z_k + z/R_k$$

and $\bar{u}_k : B(0, \varepsilon'_k R_k) \rightarrow W$ by

$$\bar{u}_k(z) = \tilde{u} \circ l_k(z) = (a(z_k + z/R_k), u(z_k + z/R_k)).$$

For any $R' > 0$, when k is large, \bar{u}_k is a J -holomorphic map defined over $B(0, R')$. Moreover, $\|\nabla \bar{u}_k(z)\| \leq 2$ for $z \in B(0, R')$. By Lemma 2, for any $n \in \mathbb{Z}^+$, there exists $C(n, R')$ satisfying

$$\|\bar{u}_k\|_{C^n(B(0, R'-1), W)} \leq C(n, R'). \quad (8)$$

We also have

$$|\nabla \bar{u}_k(0)| = 1 \quad (9)$$

$$|\nabla \bar{u}_k(z)| \leq 2 \text{ for all } |z| \leq \varepsilon'_k R_k. \quad (10)$$

Define $\tilde{u}_k(z) := (a(z_k + z/R_k) - a(z_k), u(z_k + z/R_k))$. Since $\{\tilde{u}_k(0)\}_{k \in \mathbb{Z}^+}$ is bounded, then by (8) we can apply Ascoli-Arzelà theorem to get a subsequence, still called \tilde{u}_k , satisfying $\tilde{u}_k \rightarrow \tilde{u}_\infty$ in C_{loc}^∞ , as $k \rightarrow \infty$. Here $\tilde{u}_\infty : \mathbb{C} \rightarrow W$ is a J_∞ -holomorphic satisfying

$$|\nabla \tilde{u}_\infty(0)| = 1 \quad \|\nabla \tilde{u}_\infty\| \leq 2.$$

Indeed, \tilde{u}_k satisfies

$$J_\infty(\tilde{u}_k)T\tilde{u}_k = T(\tilde{u}_k)i + L_k T\bar{u}_k i, \quad (11)$$

⁴ Actually, to prove Proposition 2, Proposition 3 and Theorem 3 we only need is $f_s^* J \rightarrow J_\infty$ in C_{loc}^1 , as $s \rightarrow \infty$. We need the stronger condition AC3) to prove exponential decay in 3.3 and thus the main theorems.

where $L_k := Tf_{-a(z_k)}(J_\infty - J)(\bar{u}_k)$. While, $\|L_k\|_{C^0(B(0, \varepsilon'_k R_k))} \rightarrow 0$, as $k \rightarrow \infty$. Therefore, \tilde{u}_∞ is J_∞ -holomorphic.

Now let's look at its energy.

$$\begin{aligned} \int_{B(0, R')} \tilde{u}_k^* \omega_\infty &= \int_{B(0, R')} \bar{u}_k^* \omega_\infty \\ &= \int_{B(0, R')} (a \circ l_k, u \circ l_k)^* \omega_\infty \end{aligned} \quad (12)$$

$$\begin{aligned} &= \int_{l_k(B(0, R'))} \tilde{u}^* \omega_\infty \\ &= \int_{B(z_k, R'/R_k)} \tilde{u}^* \omega + \int_{B(z_k, R'/R_k)} \tilde{u}^* (\omega - \omega_\infty). \end{aligned} \quad (13)$$

From $E(\tilde{u}) < +\infty$ we see

$$\int_{B(z_k, R'/R_k)} \tilde{u}^* \omega \rightarrow 0,$$

as $k \rightarrow +\infty$. While,

$$\begin{aligned} \left| \int_{B(z_k, R'/R_k)} \tilde{u}^* (\omega_\infty - \omega) \right| &\leq \int_{B(z_k, R'/R_k)} |(\omega_\infty - \omega)(\tilde{u}_s, \tilde{u}_t)| ds \wedge dt \\ &= \int_{B(z_k, R'/R_k)} (2R_k)^2 \left| (\omega_\infty - \omega) \left(\frac{\tilde{u}_s}{2R_k}, \frac{\tilde{u}_t}{2R_k} \right) \right| ds \wedge dt. \end{aligned}$$

By AC4) we have

$$c_k := \sup_{z \in B(z_k, \varepsilon'_k)} \left| (\omega_\infty - \omega) \left(\frac{\tilde{u}_s}{2R_k}, \frac{\tilde{u}_t}{2R_k} \right) \right| \rightarrow 0$$

as $k \rightarrow \infty$. Thus,

$$\left| \int_{B(z_k, R'/R_k)} \tilde{u}^* (\omega_\infty - \omega) \right| \leq \pi \left(\frac{R'}{R_k} \right) (2R_k)^2 c_k \rightarrow 0$$

as $k \rightarrow \infty$. Therefore,

$$E_{\omega_\infty}(\tilde{u}_\infty) = \int_{\mathbb{C}} \tilde{u}_\infty^* \omega_\infty = 0.$$

Moreover, we have $E_{\lambda_\infty}(\tilde{u}_\infty) < +\infty$. Indeed, given $\phi \in \mathcal{C}$, denote $\phi_k(r) := \phi(r - a(z_k)) \in \mathcal{C}$. Then we have

$$\begin{aligned}
& \left| \int_{B(0, R')} \tilde{u}_k^*(\phi(r) dr \wedge \lambda_\infty) \right| \\
&= \left| \int_{l_k[B(0, R')]} \phi_k(a) \tilde{u}^*(dr \wedge \lambda_\infty) \right| \\
&\leq \left| \int_{B(z_k, R'/R_k)} \phi_k(a) \tilde{u}^*(\sigma \wedge \lambda) \right| + \left| \int_{B(z_k, R'/R_k)} \phi_k(a) \tilde{u}^*(dr \wedge \lambda_\infty - \sigma \wedge \lambda) \right|. \quad (14)
\end{aligned}$$

While,

$$\left| \int_{B(z_k, R'/R_k)} \phi_k(a) \tilde{u}^*(\sigma \wedge \lambda) \right| \leq \left| \int_{\mathbb{C}} \phi_k(a) \tilde{u}^*(\sigma \wedge \lambda) \right| \leq E_\lambda(\tilde{u}) \quad (15)$$

and

$$\begin{aligned}
& \left| \int_{B(z_k, R'/R_k)} \phi_k(a) \tilde{u}^*(dr \wedge \lambda_\infty - \sigma \wedge \lambda) \right| \\
&\leq \int_{B(z_k, R'/R_k)} \phi_k(a) (2R_k)^2 \left| (dr \wedge \lambda_\infty - \sigma \wedge \lambda) \left(\frac{\tilde{u}_s}{2R_k}, \frac{\tilde{u}_t}{2R_k} \right) \right| ds \wedge dt. \quad (16)
\end{aligned}$$

Since $r_k := \sup_{z \in B(z_k, R'/R_k)} \left| (dr \wedge \lambda_\infty - \sigma \wedge \lambda) \left(\frac{\tilde{u}_s}{2R_k}, \frac{\tilde{u}_t}{2R_k} \right) \right| \rightarrow 0$ as $k \rightarrow \infty$, we have

$$\begin{aligned}
& \left| \int_{B(z_k, R'/R_k)} \phi_k(a) \tilde{u}^*(dr \wedge \lambda_\infty - \sigma \wedge \lambda) \right| \\
&\leq \int_{B(z_k, R'/R_k)} \phi_k(a) (2R_k)^2 r_k ds \wedge dt \\
&\leq \left(\sup_{x \in \mathbb{R}} \phi(x) \right) (2R_k)^2 r_k \pi \left(\frac{R'}{R_k} \right)^2 \rightarrow 0. \quad (17)
\end{aligned}$$

Combining (14), (15), (17), we get: given $R' > 0$ and $\phi \in \mathcal{C}$, there exists constant K such that for all $k > K$,

$$\left| \int_{B(0,R)} \tilde{u}_k^*(\phi(r)dr \wedge \lambda_\infty) \right| \leq E_\lambda(\tilde{u}) + 1.$$

Therefore, $E_{\lambda_\infty}(\tilde{u}_\infty) \leq E_\lambda(\tilde{u}) + 1$. Altogether, we get a J_∞ -holomorphic map $\tilde{u}_\infty : \mathbb{C} \rightarrow W$ satisfying

$$\begin{aligned} \|\nabla \tilde{u}_\infty\| &\leq 2 & |\nabla \tilde{u}_\infty(0)| &= 1 \\ E_{\omega_\infty}(\tilde{u}_\infty) &= 0 & E(\tilde{u}_\infty) &< +\infty. \end{aligned}$$

By Proposition 1, we get a contradiction, which finishes the proof for Case 1.

Case 2: $\{a(z_k)\}_{k \in \mathbb{Z}}$ is bounded.

Now let us define \tilde{u}_k differently from Case 1 by:

$$\tilde{u}_k(z) := \tilde{u} \circ l_k = (a(z_k + z/R_k), u(z_k + z/R_k)),$$

then \tilde{u}_k satisfies

$$\begin{aligned} |\nabla \tilde{u}_k(z)| &\leq 2 \text{ for } z \in B(0, \varepsilon_k R_k); \\ \{\tilde{u}_k(0)\}_{k \in \mathbb{Z}^+} &\text{ is bounded; } & |\nabla \tilde{u}(0)| &= 1. \end{aligned}$$

Similarly as in Case 1, by applying Ascoli-Arzelà theorem we get a subsequence still called \tilde{u}_k converging to $\tilde{u}_\infty = (a_\infty, u_\infty) : \mathbb{C} \rightarrow W$ in C_{loc}^∞ sense. Here \tilde{u}_∞ is J -holomorphic satisfying

$$|\nabla \tilde{u}_\infty(0)| = 1, \tag{18}$$

$$\|\nabla \tilde{u}_\infty\| \leq 2, \tag{19}$$

and

$$\int_{B(0, \varepsilon_k R_k)} \tilde{u}_k^* \omega = \int_{B(z_k, \varepsilon_k)} \tilde{u}^* \omega \rightarrow 0 \text{ as } k \rightarrow +\infty. \tag{20}$$

Thus, $E_\omega(\tilde{u}_\infty) = \int_{\mathbb{C}} \tilde{u}_\infty^* \omega = 0$. Moreover, given $R' > 0$ and $\phi \in \mathcal{C}$ we have

$$\int_{B(0, R')} \tilde{u}_k^* [\phi(r)\sigma \wedge \lambda] = \int_{B(z_k, R'/R_k)} \tilde{u}^* [\phi(r)\sigma \wedge \lambda] \rightarrow 0,$$

as $k \rightarrow +\infty$. This means $\int_{B(0, R')} \tilde{u}_\infty^* [\phi(r)\sigma \wedge \lambda] = 0$, so $E_\lambda(\tilde{u}_\infty) = 0$. Hence, \tilde{u}_∞ is constant, contradicting (18). \square

Let $\tilde{v} : \mathbb{R}^+ \times S^1 \rightarrow W$ be a J -holomorphic map with respect to the standard almost complex structure on $\mathbb{R}^+ \times S^1$, i.e. it solves $\tilde{v}_s + J(\tilde{v})\tilde{v}_t = 0$ where (s, t) is the coordinate for $\mathbb{R}^+ \times S^1$.

Proposition 3. *Let \tilde{v} be defined as above. Assume $E(\tilde{v}) < +\infty$, then we have*

$$\|\nabla \tilde{v}\| < +\infty,$$

where $\|\nabla \tilde{v}\| := \sup_{(s,t) \in \mathbb{R}^+ \times S^1} |\nabla \tilde{v}(s,t)|$, and the norm $|\cdot|$ is computed with respect to the standard metric $ds^2 + dt^2$ on $\mathbb{R}^+ \times S^1$ and a translationally invariant metric g_W on W , and ∇ is the Levi-Civita connection with respect to g_W .

Proof. The proof is almost the same as the proof of Proposition 2. \square

Remark 2. Actually, we can see that we can get a gradient bound with respect to a metric g_D on the domain and a translationally invariant metric g_W on W , as long as the injective radius of g_D is bounded away from 0.

3.2 Subsequence convergence

Theorem 3. *Let (W, J) be an asymptotically cylindrical almost complex manifold, $\tilde{v} : \mathbb{R}^+ \times S^1 \rightarrow W$ be a J -holomorphic curve with $E(\tilde{v}) < +\infty$, and $\tilde{v}(\mathbb{R}^+ \times S^1)$ be unbounded, where we view S^1 as \mathbb{R}/\mathbb{Z} . Then for any sequence $R_k \rightarrow +\infty$, there exists a subsequence R_{k_n} , such that $v(R_{k_n}, \cdot)$ converges in $C^\infty(S^1)$ to a map $S^1 \rightarrow V$ given by $t \mapsto x(tT)$, where x is a $|T|$ -periodic solution of $\dot{x} = \mathbf{R}_\infty(x)$.*

Proof. By Proposition 3 we have $\|\nabla \tilde{v}\| \leq C$ for some $C > 0$. Since $\tilde{v}(\mathbb{R}^+ \times S^1)$ is not bounded, there exist a sequence of points $(s_k, t_k) \in \mathbb{R}^+ \times S^1$, such that $|a(s_k, t_k)| \rightarrow +\infty$, where $\tilde{v} = (a, v)$. Now there are two cases.

Case 1: $a(s_k, t_k) \rightarrow +\infty$.

If there exist a sequence of points $(s'_k, t'_k) \in \mathbb{R}^+ \times S^1$, such that $a(s'_k, t'_k) < Q$ for some constant Q . Pick a subsequence of (s_k, t_k) , still called (s_k, t_k) , and a subsequence of (s'_k, t'_k) , still called (s'_k, t'_k) , so that they satisfy $s'_k < s_k < s'_{k+1}$ for all k . This is possible because $s_k \rightarrow +\infty$. Since $\|\nabla \tilde{v}\| \leq C$, we have $a(s'_k, t) < Q + C$ for $t \in S^1$. Consider the compact manifold $N = [Q, Q + 2C] \times M \subset W$. Pick $\phi \in \mathcal{C}$, such that $\phi|_N > 0$. By Monotonicity Lemma, there exists $\iota > 0$ such that $\int_{\tilde{v}([s'_k, s_k] \times S^1)} \omega + \phi(r)\sigma \wedge \lambda \geq \iota > 0$ for all k . This contradicts to the fact that $E(\tilde{v}) < +\infty$. Thus $a(s, t) \rightarrow +\infty$ uniformly in t as $s \rightarrow +\infty$.

Define

$$\tilde{v}_n(s, t) = (a(s + k_n, t) - a(k_n, 0), v(s + k_n, t)),$$

and then the sequence $\tilde{v}_n(0, 0) = (0, v(k_n, 0))$ is bounded. Since \tilde{v} is J -holomorphic, by Lemma 2 and Ascoli-Arzelà Theorem, there exists a subsequence still called \tilde{v}_n converging to $\tilde{v}_\infty = (b, v_\infty) : \mathbb{R} \times S^1 \rightarrow W$ in C^∞_{loc} . We know \tilde{v}_∞ is J_∞ -holomorphic. Define the translation map $\tau_n : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times S^1$ by $\tau_n(s, t) = (s + k_n, t)$. Observe

$$\begin{aligned}
\int_{[-R,R] \times S^1} \tilde{v}_n^* \omega_\infty &= \int_{[-R,R] \times S^1} (\tilde{v} \circ \tau_n)^* \omega_\infty \\
&= \int_{[-R+k_n, R+k_n] \times S^1} \tilde{v}^* \omega + \int_{[-R+k_n, R+k_n] \times S^1} \tilde{v}^* (\omega_\infty - \omega). \tag{21}
\end{aligned}$$

For the first term on the right hand side we have

$$\int_{[-R+k_n, R+k_n] \times S^1} \tilde{v}^* \omega \rightarrow 0 \tag{22}$$

as $n \rightarrow \infty$. The second term satisfies

$$\begin{aligned}
\left| \int_{[-R+k_n, R+k_n] \times S^1} \tilde{v}^* (\omega_\infty - \omega) \right| &\leq \int_{[-R+k_n, R+k_n] \times S^1} |(\omega_\infty - \omega)(\tilde{v}_s, \tilde{v}_t)| ds \wedge dt \\
&\leq \int_{[-R+k_n, R+k_n] \times S^1} C^2 \delta_n ds \wedge dt,
\end{aligned}$$

where $\delta_n := \sup_{(s,t) \in [-R+k_n, R+k_n] \times S^1} |(\omega_\infty - \omega)(\tilde{v}_s, \tilde{v}_t)| \rightarrow 0$ as $n \rightarrow +\infty$. Thus,

$$\left| \int_{[-R+k_n, R+k_n] \times S^1} \tilde{v}^* (\omega_\infty - \omega) \right| \leq 2RC^2 \delta_n \rightarrow 0 \tag{23}$$

as $n \rightarrow +\infty$. Combining (21), (22) and (23), we can see $\int_{[-R,R] \times S^1} \tilde{v}_\infty^* \omega_\infty = 0$.

Therefore, $E_{\omega_\infty}(\tilde{v}_\infty) = 0$, so there exists a smooth map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\tilde{v}_\infty = (b, x \circ f)$, where x is the solution of $\dot{x} = \mathbf{R}_\infty(x)$. Let Φ be the holomorphic function defined by $\Phi = b + if$. Since $\|\nabla \Phi\| \leq C$, Φ is linear. Thus, $\Phi(s, t) = \alpha(s + it) + \beta$, where $\alpha = T + il, \beta = m + in \in \mathbb{C}$ are constants. But $b(s, t) - b(s, t+1) = 0$ implies $l = 0$, and $b(0, 0) = 0$ implies $m = 0$. Thus,

$$f = Tt + n, \tag{24}$$

$$b = Ts. \tag{25}$$

Therefore, $a_s(k_n, t) \rightarrow T$ uniformly in t as $n \rightarrow +\infty$ (Recall the notation $\tilde{v} = (a, v)$, $\tilde{v}_\infty = (b, v_\infty)$). Moreover, we have

$$\int_{\{0\} \times S^1} \tilde{v}_\infty^* \lambda_\infty = \int_{\{0\} \times S^1} \lambda_\infty [(\tilde{v}_\infty)_t] dt = \int_{\{0\} \times S^1} b_s dt = T. \tag{26}$$

Claim: $T \neq 0$.

Under this claim, \tilde{v}_∞ is not constant. By (24), $f(s, t+1) = T(t+1) + n$, so $x(T(t+1) + n) = x(Tt + n)$. Hence, x is T -periodic.

Proof of Claim. Suppose $T = 0$. Since $a(s, t) \rightarrow +\infty$ uniformly in t as $s \rightarrow +\infty$, we can pick a subsequence k_{n_m} of k_n and a sequence $t_m \in S^1$, such that $a(k_{n_m+1}, t_{m+1}) - a(k_{n_m}, t_m) \geq 4C$. Denote $a(k_{n_m}, t_m)$ by a_m . From $\|\nabla \tilde{u}\| \leq C$ we get

$$a(k_{n_m}, t) \in [a_m - C, a_m + C], \quad (27)$$

$$a(k_{n_m+1}, t) \geq a_m + 3C. \quad (28)$$

Let $\psi_m : \mathbb{R} \rightarrow [0, 1]$ be a smooth map, satisfying $\psi_m(r) = \frac{1}{7C}(r - a_m + \frac{3}{2}C)$ for $r \in [a_m - C, a_m + 5C]$, and $\phi_m = \psi_m' \in \mathcal{C}$. We can further require $C > 1$, then $\phi_m(r) \leq \frac{1}{7C} < 1$. Observe

$$\int_{[k_{n_m}, k_{n_m+1}] \times S^1} \tilde{v}^* d(\psi_m(r)\lambda) = \int_{\{k_{n_m+1}\} \times S^1} \tilde{v}^*(\psi_m(r)\lambda) - \int_{\{k_{n_m}\} \times S^1} \tilde{v}^*(\psi_m(r)\lambda).$$

While,

$$\begin{aligned} \left| \int_{\{k_{n_m+1}\} \times S^1} \tilde{v}^*(\psi_m(r)\lambda) \right| &= \left| \int_{\{k_{n_m+1}\} \times S^1} \psi_m(\tilde{v})\lambda(\tilde{v}_t) dt \right| \\ &\leq \int_{\{k_{n_m+1}\} \times S^1} |\lambda(\tilde{v}_t)| dt \rightarrow T = 0, \end{aligned}$$

as $m \rightarrow +\infty$. Similarly, $\int_{\{k_{n_m}\} \times S^1} \tilde{v}^*(\psi_m(r)\lambda) \rightarrow 0$. Thus,

$$\int_{[k_{n_m}, k_{n_m+1}] \times S^1} \tilde{v}^* d(\psi_m(r)\lambda) \rightarrow 0. \quad (29)$$

Observe

$$\begin{aligned} &\int_{[k_{n_m}, k_{n_m+1}] \times S^1} \tilde{v}^*(\phi_m(r)\sigma \wedge \lambda) \\ &= \int_{[k_{n_m}, k_{n_m+1}] \times S^1} \tilde{v}^*(\phi_m(r)dr \wedge \lambda) + \int_{[k_{n_m}, k_{n_m+1}] \times S^1} \tilde{v}^*[\phi_m(r)(\sigma - dr) \wedge \lambda] \end{aligned} \quad (30)$$

While,

$$\begin{aligned}
& \left| \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^*(\phi_m(r) dr \wedge \lambda) \right| \\
& \leq \left| \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* d(\psi_m(r) \lambda) \right| + \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} |\tilde{v}^*(\psi_m(r) d\lambda)| \\
& \leq \left| \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* d(\psi_m(r) \lambda) \right| + \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^*(c\omega + c_m \sigma \wedge \lambda), \quad (31)
\end{aligned}$$

for some $c > 0$, $c_m > 0$. The second inequality is due to the fact that $c\omega + c_m \sigma \wedge \lambda$ is non-degenerate and tames J ; also since $d\lambda \rightarrow d\lambda_\infty$ and $i(\frac{\partial}{\partial r})d\lambda_\infty = 0 = i(\mathbf{R}_\infty)d\lambda_\infty$, we can require that c is independent of m and c_m goes to 0 as $m \rightarrow +\infty$. Similarly, we have

$$\left| \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* [\phi_m(r)(\sigma - dr) \wedge \lambda] \right| \leq \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* [c\omega + c_m \sigma \wedge \lambda]. \quad (32)$$

When k is large, from (30), (31) and (32) we get

$$\begin{aligned}
& \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^*(\phi_m(r) \sigma \wedge \lambda) \\
& \leq D \left\{ \left| \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* d(\psi_m(r) \lambda) \right| + \int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* \omega \right\}, \quad (33)
\end{aligned}$$

for some constant $D > 0$ which does not depend on m and \tilde{v} . The reason that the term $\int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^*(c_m \sigma \wedge \lambda)$ does not show up on the right hand side of (33) is because that it is absorbed by the left hand side since $\phi_m|_{[k_{n_m}, k_{n_{m+1}}] \times S^1} = 1/7$. Since $E_\omega(\tilde{v})$ is finite, $\int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^* \omega$ goes to 0. Together with (29), we get

$$\int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^*(\phi_m(r) dr \wedge \lambda) \rightarrow 0.$$

Summing up,

$$\int_{[k_{n_m}, k_{n_{m+1}}] \times S^1} \tilde{v}^*(\omega + \phi_m(r) dr \wedge \lambda) \rightarrow 0 \quad (34)$$

as $m \rightarrow +\infty$.

Now consider $N_m = [a_m + C, a_m + 3C] \times V \subseteq W$ with an almost complex structure $J_m := J|_{N_m}$ and a non-degenerate 2-form $\Omega_m := \omega + \phi_m(r)\sigma \wedge \lambda|_{N_m}$. Because of the asymptotic condition, we can find uniform constants $C_0, r_0 > 0$ such that the Monotonicity Lemma holds for all m , i.e. for any J_m -holomorphic curve $h_m : (S, j) \rightarrow (N_m, J_m)$ where (S, j) is a Riemann surface with boundary, and if the boundary $h_m(\partial S)$ is contained in the complement of the ball $B(h_m(s_0), r)$ where $s_0 \in \text{Int}S_m$ and $r < r_0$, then we have $\int_{h_m(S) \cap B(h_m(s_0), r)} \Omega_m \geq C_0 r^2$. By (27) and (28) we can see $\tilde{u}(k_{n_m}, S^1) \cap \text{Int}N_m = \emptyset$ and $\tilde{u}(k_{n_m+1}, S^1) \cap \text{Int}N_m = \emptyset$. This contradicts to (34). Thus, $T \neq 0$.

Case 2. $a(s_k, t_k) \rightarrow -\infty$.

We deal with it similarly. In this case we get $T < 0$. \square

Corollary 1. *Under the assumption of Theorem 3, as $s \rightarrow +\infty$,*

$$\partial^\beta[a(s, t) - Ts] \rightarrow 0$$

uniformly in t , provided $\beta \in \mathbb{N} \times \mathbb{N}$ and $|\beta| \geq 1$.

Proof. Suppose to the contrary, then there exists a sequence of points (s_k, t_k) such that $s_k \rightarrow +\infty$ and $\partial^\beta[a(s, t) - Ts]|_{(s_k, t_k)} \rightarrow c$ as $k \rightarrow +\infty$ for some $|\beta| \geq 1$, where c is a non-zero constant ($\pm\infty$ included). Define $\bar{a}_k(s, t) := a(s + s_k, t + t_k) - a(s_k, t_k)$, and then $\bar{a}_k(0, 0) = 0$. From the proof of Theorem 3 we get a subsequence of k , still called k , such that $\bar{a}_k \rightarrow T's$ in $C_{loc}^\infty(\mathbb{R}^+ \times S^1, \mathbb{R})$. However, from the Morse-Bott condition, we get $T' = T$. Thus,

$$\begin{aligned} \partial^\beta[a(s, t) - Ts]|_{(s_k, t_k)} &= \partial^\beta[a(s + s_k, t + t_k) - a(s_k, t_k) - Ts]|_{(0, 0)} \\ &= \partial^\beta(\bar{a}_k(s, t) - Ts)|_{(0, 0)} \\ &\rightarrow 0, \end{aligned}$$

which contradicts to the assumption. \square

To prove Theorem 1 and Theorem 2, we need to get the exponential decay estimates.

3.3 Exponential decay

In this subsection, we will follow the schemes in [4] to prove Theorem 1 and Theorem 2. The strategy is as follows: Firstly, we pick a neighborhood of the orbit γ , restrict the J -holomorphic curve to a sequence of cylinders inside the domain so that the images lie in the neighborhood and satisfy certain inequalities, and estimate the behaviors of each cylinder by the behaviors of boundaries of the cylinder. Secondly, we show that near the infinity end of the domain the J -holomorphic curve satisfies the inequalities and lies in the neighborhood of γ based on the estimates. Once these are achieved, Theorem 1 and Theorem 2 follow automatically.

In order to study the J -holomorphic curve equation around γ , we need introduce a good coordinate chart around a neighborhood of γ .

Lemma 3. [5] *Suppose that $(W = \mathbb{R} \times V, J)$ is an asymptotically cylindrical almost complex manifold. Let N be a component of the set $N_T \subset V$ (see Definition 4), and γ be one of the orbits from N .*

a) if T is the minimal period of γ then there exists a neighborhood $U \supset \gamma$ in V such that $U \cap N$ is invariant under the flow of \mathbf{R}_∞ and one finds coordinates $(\vartheta, x_1, \dots, x_n, y_1, \dots, y_n)$ of U such that

$$N = \{x_1, \dots, x_p = 0, y_1, \dots, y_q = 0\},$$

for $0 \leq p, q \leq n$,

$$\mathbf{R}_\infty|_N = \frac{\partial}{\partial \vartheta},$$

and

$$\omega|_N = \omega_0|_N,$$

where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

b) if γ is a m -multiple of a trajectory x of a minimal period $\frac{T}{m}$ there exists a tubular neighborhood \tilde{U} of γ such that its m -multiple cover U together with all the structures induced by the covering map from $U \rightarrow \tilde{U}$ from the corresponding objects on \tilde{U} satisfy the properties of the part a).

Proof. Refer to Lemma A.1 in [5]. □

Using this coordinate chart, we can work locally in $U \subset (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^{2n}$, and make T the minimal period of γ . Denote by z_{in} the coordinate $(x_1, \dots, x_p, y_1, \dots, y_q)$ and by z_{out} the coordinate $(x_{n-p+1}, \dots, x_n, y_{n-p+1}, \dots, y_n)$. We can easily see following lemma about behavior of a J -holomorphic curve in the z_{out} direction.

Lemma 4. *Let J be an asymptotically cylindrical almost complex structure on $W = \mathbb{R} \times V$, and \tilde{u} be a finite energy J -holomorphic curve from $\mathbb{R}^+ \times S^1$ to W . Suppose $[m_k, n_k]$ is a sequence of intervals in \mathbb{R}^+ with $m_k \rightarrow +\infty$ and $\tilde{u}([m_k, n_k] \times S^1) \subset U$, then we have as $k \rightarrow +\infty$,*

$$\sup_{(s,t) \in [m_k, n_k] \times S^1} |\partial^\beta z_{out}(s, t)| \rightarrow 0$$

for all $\beta \in \mathbb{N} \times \mathbb{N}$.

Proof. The proof is very similar to the proof of Corollary 1, so we omit it here. □

Let's study the J -holomorphic curve equation in $\mathbb{R} \times U \subset \mathbb{R} \times (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^{2n}$. Denote $\theta := [s_0, s_1] \times S^1$ for some $s_0 < s_1$ and let $\tilde{u} = (a, \vartheta, z) : \theta \rightarrow \mathbb{R} \times U$ be a J -holomorphic curve, then we have

$$\tilde{u}_s + J\tilde{u}_t = 0,$$

i.e.

$$(a_s, \vartheta_s, z_s) + J(a_t, \vartheta_t, z_t) = 0. \quad (35)$$

The z component of left hand side of this equation is

$$z_s + \pi_z J|_z z_t + a_t \pi_z \mathbf{R} + \vartheta_t \pi_z J|_{\vartheta} \frac{\partial}{\partial \vartheta}, \quad (36)$$

where π_z is the projection to z component using the Euclidean metric in $\mathbb{R} \times (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^{2n}$. Let's introduce the following notations:

$$\begin{aligned} & \pi_z J|_{\vartheta} \frac{\partial}{\partial \vartheta} (a, \vartheta, z_{in}, z_{out}) - \pi_z J|_{\vartheta} \frac{\partial}{\partial \vartheta} (a, \vartheta, z_{in}, 0) \\ &= \left[\int_0^1 \frac{\partial (\pi_z J|_{\vartheta} \frac{\partial}{\partial \vartheta})}{\partial z_{out}} (a, \vartheta, z_{in}, \tau z_{out}) d\tau \right] \cdot z_{out} \\ &=: S_1(a, \vartheta, z) \cdot z_{out}, \end{aligned}$$

$$\begin{aligned} & \pi_z \mathbf{R}(a, \vartheta, z_{in}, z_{out}) - \pi_z \mathbf{R}(a, \vartheta, z_{in}, 0) \\ &= \left[\int_0^1 \frac{\partial (\pi_z \mathbf{R})}{\partial z_{out}} (a, \vartheta, z_{in}, \tau z_{out}) d\tau \right] \cdot z_{out} \\ &=: S_2(a, \vartheta, z) \cdot z_{out}, \end{aligned}$$

$$S(s, t) := \vartheta_t(s, t) S_1(\tilde{u}(s, t)) + a_t(s, t) S_2(\tilde{u}(s, t)),$$

$$L(s, t) := a_t \pi_z \mathbf{R}(a, \vartheta, z_{in}, 0) + \vartheta_t \pi_z J|_{\vartheta} \frac{\partial}{\partial \vartheta} (a, \vartheta, z_{in}, 0),$$

$$M(s, t) := \pi_z J|_z.$$

With these notations (36) becomes

$$z_s + M z_t + S z_{out} + L.$$

The a component of left hand side of (35) is

$$\begin{aligned} & a_s + a_t \pi_a J \frac{\partial}{\partial a} + \vartheta_t \pi_a J \frac{\partial}{\partial \vartheta} + \pi_a J|_z z_t \\ &= a_s + a_t \pi_a \mathbf{R} + \vartheta_t \pi_a J \mathbf{R} + \vartheta_t \pi_a J \left(\frac{\partial}{\partial \vartheta} - \mathbf{R} \right) + \pi_a J|_z z_t \\ &= a_s - \vartheta_t + a_t \pi_a \mathbf{R} + \vartheta_t \pi_a J \left(\frac{\partial}{\partial \vartheta} - \mathbf{R} \right) + \pi_a J|_z z_t, \end{aligned} \quad (37)$$

Denote

$$\begin{aligned} N_1(a, \vartheta, z_{in}, z_{out}) &:= a_t \pi_a \mathbf{R}, \\ N &:= N_1(a, \vartheta, z_{in}, z_{out}) + N_2(a, \vartheta, z_{in}, 0), \\ B &:= \left(\int_0^1 \frac{\partial}{\partial z_{out}} N_2(a, \vartheta, z_{in}, \tau z_{out}) d\tau \right), \end{aligned}$$

$$B' := \pi_a J|_z.$$

Hence,

$$a_s - \vartheta_t + Bz_{out} + B'z_t + N = 0. \quad (38)$$

Apply J to (35) we get, $J(a_s, \vartheta_s, z_s) - (a_t, \vartheta_t, z_t) = 0$, then the a component of its left hand side equals

$$\begin{aligned} & -a_t + \vartheta_s \pi_a J \frac{\partial}{\partial \vartheta} + \pi_a J|_z z_s + a_s \pi_a J|_a \frac{\partial}{\partial a} \\ &= -a_t + \vartheta_s \pi_a J \mathbf{R} + \pi_a J|_z z_s + a_s \pi_a \mathbf{R} - \vartheta_s \pi_a J(\mathbf{R} - \frac{\partial}{\partial \vartheta}) \\ &= -a_t - \vartheta_s + B'z_s + a_s \pi_a \mathbf{R} - \vartheta_s \pi_a J(\mathbf{R} - \frac{\partial}{\partial \vartheta}), \end{aligned}$$

Denote

$$\begin{aligned} O_1 &:= \vartheta_s \pi_a J(\mathbf{R} - \frac{\partial}{\partial \vartheta}), \\ C &:= \left[\int_0^1 \frac{\partial}{\partial z_{out}} O_1(a, \vartheta, z_{in}, \tau z_{out}) d\tau \right], \\ O &:= -a_s \pi_a \mathbf{R} + O_1(a, \vartheta, z_{in}, 0), \\ C' &:= -B'. \end{aligned}$$

Therefore,

$$a_t + \vartheta_s + Cz_{out} + C'z_s + O = 0. \quad (39)$$

Altogether, we have⁵

$$z_s + Mz_t + Sz_{out} + L = 0, \quad (40)$$

$$a_s - \vartheta_t + Bz_{out} + B'z_t + N = 0, \quad (41)$$

$$a_t + \vartheta_s + Cz_{out} + C'z_s + O = 0. \quad (42)$$

Define an operator $A(s) : W^{1,2}(S^1, \mathbb{R}^{2n}) \rightarrow L^2(S^1, \mathbb{R}^{2n})$ by

$$(A(s)w)(t) = -M(\tilde{u}(s, t))w_t(t) - S(\tilde{u}(s, t))w_{out}(t),$$

then by (40) we get

$$A(s)z(s, \cdot) = z_s + L. \quad (43)$$

Notice that $A(s)$ depends on the map $\tilde{u} = (a, \vartheta, z_{in}, z_{out})$. If we don't use the original J -holomorphic curve \tilde{u} , instead, we substitute $\vartheta(s, t) = \vartheta(s_0, 0) + Tt$, $a(s, t) = Ts$, $z_{out}(s, t) = 0$, and $z_{in}(s, t) = z_{in}(s_0, t)$, then we get another

⁵From (40) we can see that if we require z , z_s and z_t decay exponentially, L has to decay exponentially. The condition $f_s^* J \rightarrow J_\infty$ in C_{loc}^∞ is not enough to guarantee that L decays exponentially fast.

operator denoted by $\tilde{A}(s)$. We can easily see that the $\lim_{s \rightarrow +\infty} \tilde{A}(s)$ exists and denote the limit operator by A_0 , similarly we get two matrices $M_0(t)$ and $S_0(t)$, and we have

$$M_0(t)^2 = -id,$$

and

$$(A_0 w)(t) = -M_0(t)w_t(t) - S_0(t)w_{out}. \quad (44)$$

Consider an inner product on $L^2(S^1, \mathbb{R}^{2n})$ given by

$$\langle u, v \rangle_0 = \int_0^1 \langle u, -J_0 M_0 v \rangle dt, \quad (45)$$

where the inner product is given by $\langle \cdot, \cdot \rangle = \omega_0(\cdot, J_0 \cdot)$. With respect to the inner product $\langle \cdot, \cdot \rangle_0$, it is easy to see that M_0 is anti-symmetric.

Lemma 5. A_0 is self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_0$.

Proof. By Lemma 6 and Lemma 7, this is evident. \square

Lemma 6. $\frac{1}{T} M_0(t) S_0(t) = \frac{\partial(\pi_{\mathbf{z}} \mathbf{R}_{\infty})}{\partial z_{out}}(v_0)$, where $v_0 = (\vartheta(s_0, 0) + t, z_{in}(s_0, t), 0)$.

Proof. In $\mathbb{R} \times U \subset \mathbb{R} \times (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^{2n}$,

$$\begin{aligned} & \frac{1}{T} M_0(t) S_0(t) \\ &= \lim_{s \rightarrow +\infty} \left[\frac{\partial}{\partial z_{out}} \left(\pi_z J|_z \cdot \pi_z J|_{\vartheta} \cdot \frac{\partial}{\partial \vartheta} \right) \right] (Ts, v_0) \\ &= - \lim_{s \rightarrow +\infty} \left[\frac{\partial}{\partial z_{out}} \left(\pi_z J|_a \cdot \pi_a J|_{\vartheta} \cdot \frac{\partial}{\partial \vartheta} + \pi_z J|_{\vartheta} \cdot \pi_{\vartheta} J|_{\vartheta} \cdot \frac{\partial}{\partial \vartheta} \right) \right] (Ts, v_0) \\ &= - \frac{\partial}{\partial z_{out}} \left[\lim_{s \rightarrow +\infty} \left(\pi_z J|_a \cdot \pi_a J|_{\vartheta} \cdot \frac{\partial}{\partial \vartheta} + \pi_z J|_{\vartheta} \cdot \pi_{\vartheta} J|_{\vartheta} \cdot \frac{\partial}{\partial \vartheta} \right) (Ts, \vartheta, z) \right] \\ &= \lim_{s \rightarrow +\infty} \frac{\partial}{\partial z_{out}} (\pi_z J|_a) \cdot \left(\pi_a J|_{\vartheta} \cdot \frac{\partial}{\partial \vartheta} \right) (Ts, v_0) \\ &= \frac{\partial(\pi_{\mathbf{z}} \mathbf{R}_{\infty})}{\partial z_{out}}(v_0). \end{aligned}$$

The second equality follows from the fact $J^2 = -id$. \square

Lemma 7.

$$\omega_0 \left(\frac{\partial(\pi_{\mathbf{z}} \mathbf{R}_{\infty})}{\partial z_{out}} u_{out}, v \right) = \omega_0 \left(u, \frac{\partial(\pi_{\mathbf{z}} \mathbf{R}_{\infty})}{\partial z_{out}} v_{out} \right)$$

for $u, v \in T_p(S^1 \times \mathbb{R}^{2n})$ satisfying $\pi_{\vartheta} u = 0 = \pi_{\vartheta} v$, where $p := (\vartheta(s_0, 0) + t, z_{in}(s_0, t), 0)$.

Proof. Let ϕ^t be the flow of \mathbf{R}_∞ . Since the flow preserves ω , we get

$$(\phi^t)^*\omega = \omega, \quad (46)$$

i.e. we have

$$\omega(d\phi^t(u), d\phi^t(v)) = \omega(u, v). \quad (47)$$

Since $\omega|_N = \omega_0|_N$, we get

$$\omega_0(d\phi^t(u), d\phi^t(v)) = \omega_0(u, v). \quad (48)$$

Differentiating both sides with respect to t using the flat connection gives us

$$\nabla_t \omega_0(d\phi^t(u), d\phi^t(v)) = 0, \quad (49)$$

i.e.

$$\omega_0(\nabla_t[d\phi^t(u)]|_{t=0}, v) + \omega_0(u, \nabla_t[d\phi^t(v)]|_{t=0}) = 0. \quad (50)$$

Let $\alpha : \mathbb{R} \rightarrow S^1 \times \mathbb{R}^{2n}$ such that $\alpha(0) = p$ and $\alpha'(0) = u$. Then

$$\begin{aligned} \nabla_t[d\phi^t(u)] &= \nabla_t[\nabla_s \phi^t(\alpha(s))]|_{s=0} = \nabla_s[\nabla_t \phi^t(\alpha(s))]|_{s=0} \\ &= \nabla_s\{\mathbf{R}_\infty[\alpha(s)]\}|_{s=0} = \nabla \mathbf{R}_\infty u = \nabla(\pi_z \mathbf{R}_\infty) u_{out}. \end{aligned}$$

□

Remark 3. A_0 is injective iff γ is non-degenerate.

It is not hard to see that $\ker A_0$ consists of the constant vector fields in N along γ_0 . Let's denote by P_0 the projection onto $\ker A_0$ with respect to $\langle \cdot, \cdot \rangle_0$, and let $Q_0 := I - P_0$. It is easy to check that Q_0 satisfies:

Lemma 8. $(Q_0 w)_t = w_t$, $(Q_0 w)_s = Q_0 w_s$, $(Q_0 w)_{out} = w_{out}$ and $Q_0 A_0 = A_0 Q_0$.

The following lemma will be needed in proving Lemma 10.

Lemma 9. *There exists a constant $C > 0$ such that*

$$\|A_0 Q_0 w\|_0 \geq C(\|Q_0 w\|_0 + \|(Q_0 w)_t\|_0)$$

for $w \in W^{1,2}(S^1, \mathbb{R}^{2n})$, where the norm $\|\cdot\|_0$ is defined using the inner product $\langle \cdot, \cdot \rangle_0$.

Proof. To prove the lemma we only need to prove $\|A_0 Q_0 w\|_0 \geq C'\|Q_0 w\|_0$ for some $C' > 0$, because by definition we have

$$A_0 Q_0 w = -M_0(Q_0 w)_t - S_0 Q_0 w. \quad (51)$$

Suppose to the contrary, there exists $\varepsilon_n \rightarrow 0$ and $w_n \in W^{1,2}(S^1, \mathbb{R}^{2n})$ satisfying $\|Q_0 w_n\|_0 = 1$ and $\|A_0 Q_0 w_n\|_0 \leq \varepsilon_n$. Then we have

$$\|(Q_0 w_n)_t\|_0 \leq \|M_0 A_0 Q_0 w_n\|_0 + \|M_0 S_0 Q_0 w_n\|_0 \leq \varepsilon_n + C''.$$

Therefore, $Q_0 w_n$ is bounded in $W^{1,2}(S^1, \mathbb{R}^{2n})$. Since $W^{1,2}(S^1, \mathbb{R}^{2n})$ embeds compactly in $L^2(S^1, \mathbb{R}^{2n})$ we get a subsequence of w_n , still denoted by w_n , such that $Q_0 w_n$ is a Cauchy sequence in $L^2(S^1, \mathbb{R}^{2n})$. But it is easy to see that $(Q_0 w_n)_t$ is also a Cauchy sequence in $L^2(S^1, \mathbb{R}^{2n})$. Therefore, $Q_0 w_n$ converges to some η in $W^{1,2}(S^1, \mathbb{R}^{2n})$, so $\eta \in \ker A_0$. Because η also lies in the orthogonal complement of $\ker A_0$, η has to be 0, which contradicts to the fact $\|\eta\|_0 = \lim_{n \rightarrow 0} \|Q_0 w_n\|_0 = 1$. \square

Denote $g_0(s) := \frac{1}{2} \|Q_0 z(s)\|_0^2$ and $\kappa_0(s) := (\vartheta(s_0, 0) - \vartheta(s, 0), z_{in}(s_0, 0) - z_{in}(s, 0))$, and then we have

Lemma 10. *There exist $\delta > 0$, $b > 0$ and $\bar{\kappa} > 0$ such that, if*

$$\begin{aligned} a(s_0, 0) &\geq b, \\ |\kappa_0(s_0)| &\leq \bar{\kappa}, \\ \sup_{(s,t) \in \theta} |\partial^\beta z_{out}(s, t)| &\leq \delta, \end{aligned}$$

for multi-indices β with $|\beta| \leq 2$, and

$$\begin{aligned} \sup_{(s,t) \in \theta} |\partial^\beta (a(s, t) - Ts)| &\leq \delta, \\ \sup_{(s,t) \in \theta} |\partial^\beta (\vartheta(s, t) - Tt)| &\leq \delta, \\ \sup_{(s,t) \in \theta} |\partial^\beta z_{in}(s, t)| &\leq \delta, \end{aligned}$$

for those multi-indices β with $0 < |\beta| \leq 2$, then for $s \in [s_0, \mathfrak{s}]$, we have

$$g_0''(s) \geq c^2 g_0(s) - c_2 e^{-c_1(s-s_0)},$$

where

$$\mathfrak{s} := \sup \{s \in [s_0, s_1] \mid |\kappa_0(s')| \leq \bar{\kappa} \text{ for all } s' \in [s_0, s]\},$$

and $c, c_1, c_2 > 0$ are constants independent of s_0 and s_1 .

Proof. Notice that from the assumption we have

$$\begin{aligned} \sup_{(s,t) \in \theta} |\partial^\beta (\vartheta(s, t) - \vartheta(s, 0) - Tt)| &\leq \delta, \\ \sup_{(s,t) \in \theta} |\partial^\beta (z_{in}(s, t) - z_{in}(s, 0))| &\leq \delta, \end{aligned}$$

for multi-indices β with $|\beta| \leq 1$.

Define an operator $\bar{A}(s) : W^{1,2}(S^1, \mathbb{R}^{2n}) \rightarrow L^2(S^1, \mathbb{R}^{2n})$ by

$$\begin{aligned} (\bar{A}(s)w)(t) &= - \lim_{a \rightarrow +\infty} M(a, \vartheta(s, t), z(s, t))w_t(t) \\ &\quad - \lim_{a \rightarrow +\infty} \vartheta_t(s, t)S_1(a, \vartheta(s, t), z(s, t))w_{out}(t). \end{aligned}$$

From (43) we get

$$z_s = A_0 z + (\Delta_0 + \tilde{\Delta}_0 \kappa_0) z_t + (\hat{\Delta}_0 + \bar{\Delta}_0 \kappa_0) z_{out} + [A(s) - \bar{A}(s)] z - L. \quad (52)$$

Applying Q_0 to (52) gives us

$$\begin{aligned} (Q_0 z)_s &= A_0 Q_0 z + Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_t + Q_0 (\hat{\Delta}_0 + \bar{\Delta}_0 \kappa_0) (Q_0 z)_{out} \\ &\quad + Q_0 [A(s) - \bar{A}(s)] z - Q_0 L, \end{aligned} \quad (53)$$

where

$$\Delta_0 = \lim_{a \rightarrow +\infty} [M(a, \vartheta(s, 0) + Tt, z_{in}(s, 0), 0) - M(a, \vartheta(s, t), z(s, t))],$$

$$\hat{\Delta}_0 = \lim_{a \rightarrow +\infty} [TS_1(a, \vartheta(s, 0) + Tt, z_{in}(s, 0), 0) - \vartheta_t(s, t) S_1(a, \vartheta(s, t), z(s, t))],$$

satisfying

$$\sup_{(s,t) \in \theta} |\partial^\beta \Delta_0(s, t)| \leq C\delta,$$

$$\sup_{(s,t) \in \theta} |\partial^\beta \hat{\Delta}_0(s, t)| \leq C\delta,$$

for multi-indices β with $|\beta| \leq 1$, and

$$\tilde{\Delta}_0 \kappa_0 = M_0 - \lim_{a \rightarrow +\infty} M(a, \vartheta(s, 0) + Tt, z_{in}(s, 0), 0),$$

$$\bar{\Delta}_0 \kappa_0 = S_0 - \lim_{a \rightarrow +\infty} TS_1(a, \vartheta(s, 0) + Tt, z_{in}(s, 0), 0),$$

satisfying

$$\sup_{(s,t) \in \theta} |\partial^\beta \tilde{\Delta}_0(s, t)| \leq C,$$

$$\sup_{(s,t) \in \theta} |\partial^\beta \bar{\Delta}_0(s, t)| \leq C,$$

for multi-indices β with $|\beta| \leq 1$. We can require $0 < \delta < \frac{T}{2}$, and we get

$$a(s, t) \geq a(s_0, 0) + \frac{T}{2}(s - s_0) - \delta \geq (b - \delta) + \frac{T}{2}(s - s_0).$$

Because J is an asymptotically cylindrical almost complex structure, we get

$$\|Q_0 L\|_0 \leq c_0 e^{-c'_0(b-\delta)} e^{-c'_0 \frac{T}{2}(s-s_0)}$$

for some constants $c_0, c'_0 > 0$. Denote $c_1 := c'_0 \frac{T}{2}$ and $c_2 := c_0 e^{-c'_0(b-\delta)}$, and then we have

$$\|Q_0 L\|_0 \leq c_2 e^{-c_1(s-s_0)}.$$

Also, from

$$\begin{aligned}
& [A(s) - \bar{A}(s)]z \\
&= \left[\lim_{a \rightarrow +\infty} M(a, \vartheta(s, t), z(s, t)) - M(a(s, t), \vartheta(s, t), z(s, t)) \right] (Q_0 z)_t \\
&+ \left[\lim_{a \rightarrow +\infty} \vartheta_t(s, t) S_1(a, \vartheta(s, t), z(s, t)) - S(a(s, t), \vartheta(s, t), z(s, t)) \right] (Q_0 z)_{out}
\end{aligned}$$

we get

$$\|\{\partial^\beta [A(s) - \bar{A}(s)]\} z\|_0 \leq c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}} \quad (54)$$

for multi-indices β with $|\beta| \leq 1$, by picking c_0 larger if necessary.

Now we are ready to estimate $g_0''(s)$. Obviously we have

$$g_0''(s) \geq \langle Q_0 z_{ss}, Q_0 z \rangle_0.$$

Now let's compute the right hand side of the above inequality. Differentiate (53) with respect to s , we obtain

$$\begin{aligned}
(Q_0 z)_{ss} &= A_0 Q_0 z_s + Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_{st} + Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0)_s (Q_0 z)_t \\
&+ Q_0 (\hat{\Delta}_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z_s)_{out} + Q_0 (\hat{\Delta}_0 + \tilde{\Delta}_0 \kappa_0)_s (Q_0 z)_{out} \\
&+ Q_0 [A(s) - \bar{A}(s)]_s z + Q_0 [A(s) - \bar{A}(s)] z_s - Q_0 L_s,
\end{aligned}$$

Thus we get $\langle Q_0 z_{ss}, Q_0 z \rangle_0$ contains 8 terms. When we are estimating these terms, each time we see $Q_0 z_s$, we replace it using (53).

$$\begin{aligned}
T_1 &= \langle A_0 Q_0 z_s, Q_0 z \rangle_0 \\
&= \left\langle A_0 Q_0 z + Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_t + Q_0 (\hat{\Delta}_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_{out} \right. \\
&\quad \left. + Q_0 [A(s) - \bar{A}(s)] z - Q_0 L, A_0 Q_0 z \right\rangle_0 \\
&\geq \|A_0 Q_0 z\|_0^2 - C(\delta + |\kappa_0|) \|Q_0 z\|_{0, W^{1,2}} \|A_0 Q_0 z\|_0 \\
&\quad - c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}} \|A_0 Q_0 z\|_0 - c_2 e^{-c_1(s-s_0)} \|A_0 Q_0 z\|_0,
\end{aligned}$$

$$\begin{aligned}
T_2 &= \left\langle Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_{st}, Q_0 z \right\rangle_0 \\
&= \int_0^1 \omega_0(Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_{st}, M_0 Q_0 z) dt \\
&= - \int_0^1 \omega_0(Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0)_t (Q_0 z)_s, M_0 Q_0 z) dt \\
&\quad - \int_0^1 \omega_0(Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_s, (M_0)_t Q_0 z) dt \\
&\quad - \int_0^1 \omega_0(Q_0 (\Delta_0 + \tilde{\Delta}_0 \kappa_0) (Q_0 z)_s, M_0 Q_0 z_t) dt \\
&\geq -3C(\delta + |\kappa_0|) \|Q_0 z_s\|_0 \|Q_0 z\|_{0, W^{1,2}} \\
&\geq -3C(\delta + |\kappa_0|) \|A_0 Q_0 z\|_0 \|Q_0 z\|_{0, W^{1,2}} - 6C(\delta + |\kappa_0|)^2 \|Q_0 z\|_{0, W^{1,2}}^2 \\
&\quad - 3C(\delta + |\kappa_0|) c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}^2 - 3C(\delta + |\kappa_0|) c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}},
\end{aligned}$$

$$T_3 = \left\langle Q_0(\Delta_0 + \bar{\Delta}_0 \kappa_0)_s(Q_0 z)_t, Q_0 z \right\rangle_0 \geq -C(\delta + |\kappa_0|) \|Q_0 z\|_{0, W^{1,2}}^2,$$

$$\begin{aligned} T_4 &= \left\langle Q_0(\hat{\Delta}_0 + \bar{\Delta}_0 \kappa_0)(Q_0 z_s)_{out}, Q_0 z \right\rangle_0 \\ &\geq -C(\delta + |\kappa_0|) \|Q_0 z_s\|_0 \|Q_0 z\|_{0, W^{1,2}} \\ &\geq -C(\delta + |\kappa_0|) \|A_0 Q_0 z\|_0 \|Q_0 z\|_{0, W^{1,2}} - 2C^2(\delta + |\kappa_0|)^2 \|Q_0 z\|_{0, W^{1,2}}^2 \\ &\quad - C(\delta + |\kappa_0|) c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}^2 - C(\delta + |\kappa_0|) c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}, \end{aligned}$$

$$T_5 = \left\langle Q_0(\hat{\Delta}_0 + \bar{\Delta}_0 \kappa_0)_s(Q_0 z)_{out}, Q_0 z \right\rangle_0 \geq -C(\delta + |\kappa_0|) \|Q_0 z\|_{0, W^{1,2}}^2,$$

$$T_6 = \langle Q_0[A(s) - \bar{A}(s)]_s z, Q_0 z \rangle_0 \geq -c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}^2,$$

$$\begin{aligned} T_7 &= \langle Q_0[A(s) - \bar{A}(s)]z_s, Q_0 z \rangle_0 \\ &\geq -c_2 e^{-c_1(s-s_0)} \|Q_0 z_s\|_{0, W^{1,2}} \|Q_0 z\|_{0, W^{1,2}} \\ &\geq -c_2 e^{-c_1(s-s_0)} \|A_0 Q_0 z\|_0 \|Q_0 z\|_{0, W^{1,2}} - 2C(\delta + |\kappa_0|) c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}^2 \\ &\quad - c_2^2 e^{-2c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}^2 - c_2^2 e^{-2c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}, \end{aligned}$$

$$T_8 = \langle -Q_0 L_s, Q_0 z \rangle_0 \geq -c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}.$$

Using all the above estimates of T_1, \dots, T_8 , Lemma 9 and the fact that

$$-c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}} \geq -c_2 e^{-c_1(s-s_0)} - c_2 e^{-c_1(s-s_0)} \|Q_0 z\|_{0, W^{1,2}}^2$$

we get

$$g_0''(s) \geq (1 - 10C\delta - 10C|\kappa_0| - 10Cc_2 e^{-c_1(s-s_0)})g_0(s) - c_2 e^{-c_1(s-s_0)}.$$

From the definition of c_2 we can see that if b is large enough, c_2 can be very close to 0. Therefore,

$$g_0''(s) \geq c^2 g_0(s) - c_2 e^{-c_1(s-s_0)}.$$

We can require further that $c_1 > c > 0$. □

Based on Lemma 10, we could easily get

Lemma 11. *Under the same assumption as in Lemma 10, we have*

$$g_0(s) \leq \max\{g_0(s_0), g_0(\mathfrak{s})\} \frac{\cosh\left[c\left(s - \frac{s_0 + \mathfrak{s}}{2}\right)\right]}{\cosh\left(c\frac{\mathfrak{s} - s_0}{2}\right)} + \frac{c_2}{c_1^2 - c^2} \frac{\sinh(c(\mathfrak{s} - s))}{\sinh(c(\mathfrak{s} - s_0))},$$

for $s_0 \leq s \leq \mathfrak{s}$.

Proof. Let

$$\begin{aligned} h(s) := & \max\{g_0(s_0), g_0(\mathfrak{s})\} \frac{\cosh\left[c\left(s - \frac{s_0 + \mathfrak{s}}{2}\right)\right]}{\cosh\left(c\frac{\mathfrak{s} - s_0}{2}\right)} \\ & + \frac{c_2}{c_1^2 - c^2} \frac{1}{\sinh(c(\mathfrak{s} - s_0))} \left\{ \sinh(c(\mathfrak{s} - s)) \right. \\ & \left. + e^{-c_1(\mathfrak{s} - s_0)} \sinh(c(s - s_0)) - e^{-c_1(s - s_0)} \sinh(c(\mathfrak{s} - s_0)) \right\}, \end{aligned}$$

then $h(s)$ satisfies:

$$\begin{cases} h''(s) - c^2 h(s) = -c_2 e^{-c_1(s - s_0)} \\ h(s_0) = \max\{g_0(s_0), g_0(\mathfrak{s})\} \\ h(\mathfrak{s}) = \max\{g_0(s_0), g_0(\mathfrak{s})\} \end{cases} \quad (55)$$

Let $l(s) := g_0(s) - h(s)$, then $l(s)$ satisfies

$$\begin{cases} l''(s) - c^2 l(s) \geq 0 \\ l(s_0) \leq 0 \\ l(\mathfrak{s}) \leq 0 \end{cases} \quad (56)$$

Then by Maximal principle we get $l(s) \leq 0$ for $s_0 \leq s \leq \mathfrak{s}$. Then the lemma follows from the fact that

$$e^{-c_1(\mathfrak{s} - s_0)} \sinh(c(s - s_0)) - e^{-c_1(s - s_0)} \sinh(c(\mathfrak{s} - s_0)) \leq 0.$$

□

Now let's study the component z_{in} .

Lemma 12. *Let e be a unit vector in \mathbb{R}^{2n} with $e_{out} = 0$. Under the assumption of Lemma 10 and for $s \in [s_0, \mathfrak{s}]$, we have*

$$|\langle z(s), e \rangle_0 - \langle z(s_0), e \rangle_0| \leq \frac{8C}{c} \max(\|Q_0 z(s_0)\|_0, \|Q_0 z(\mathfrak{s})\|_0) + d \cdot \sqrt{c_2},$$

where $d = \frac{16C}{c} \sqrt{\frac{2}{c_1^2 - c^2}} + \frac{\sqrt{c_2}}{c_1}$, and C is a constant independent of s_0 and s_1 .

Proof. The inner product of the Cauchy-Riemann equation (40) with e gives

$$\frac{d}{ds} \langle z, e \rangle_0 + \langle M z_t, e \rangle_0 + \langle S z_{out}, e \rangle_0 + \langle L, e \rangle_0 = 0.$$

From

$$\begin{aligned} \langle M z_t, e \rangle_0 &= \int_0^1 \omega_0(M(Q_0 z)_t, M_0 e) dt \\ &= - \int_0^1 \omega_0(M_t Q_0 z, M_0 e) dt - \int_0^1 \omega_0(M Q_0 z, (M_0)_t e) dt \end{aligned}$$

we can see

$$|\langle Mz_t, e \rangle_0| \leq C \|Q_0 z\|_0.$$

Together with the facts $|\langle Sz_{out}, e \rangle_0| \leq C \|Q_0 z\|_0$ and $|\langle L, e \rangle_0| \leq c_2 e^{-c_1(s-s_0)}$ we get

$$\begin{aligned} \langle z(s), e \rangle_0 - \langle z(s_0), e \rangle_0 &\leq \int_{s_0}^s \left[2C \|Q_0 z(\mathfrak{x})\|_0 + c_2 e^{-c_1(\mathfrak{x}-s_0)} \right] d\mathfrak{x} \\ &\leq 2C \int_{s_0}^s \sqrt{2g_0(\mathfrak{x})} d\mathfrak{x} + \frac{c_2}{c_1}. \end{aligned}$$

While, Lemma 10 gives us

$$\begin{aligned} \int_{s_0}^s \sqrt{2g_0(\mathfrak{x})} d\mathfrak{x} &\leq \sqrt{\frac{2 \max\{g_0(s_0), g_0(\mathfrak{s})\}}{\cosh\left(c \frac{\mathfrak{s}-s_0}{2}\right)}} \int_{s_0}^s \sqrt{\cosh\left[c \left(\mathfrak{x} - \frac{s_0 + \mathfrak{s}}{2}\right)\right]} d\mathfrak{x} \\ &\quad + \sqrt{\frac{2c_2}{c_1^2 - c^2} \frac{1}{\sinh(c(\mathfrak{s} - s_0))}} \int_{s_0}^s \sqrt{\sinh(c(\mathfrak{s} - \mathfrak{x}))} d\mathfrak{x}. \end{aligned}$$

For the first integral we have

$$\begin{aligned} &\sqrt{\frac{2 \max\{g_0(s_0), g_0(\mathfrak{s})\}}{\cosh\left(c \frac{\mathfrak{s}-s_0}{2}\right)}} \int_{s_0}^s \sqrt{\cosh\left[c \left(\mathfrak{x} - \frac{s_0 + \mathfrak{s}}{2}\right)\right]} d\mathfrak{x} \\ &\leq \sqrt{\frac{2 \max\{g_0(s_0), g_0(\mathfrak{s})\}}{\cosh\left(c \frac{\mathfrak{s}-s_0}{2}\right)}} \left[\frac{2\sqrt{2}}{c} \sinh\left(\frac{c}{2} \left(s - \frac{s_0 + \mathfrak{s}}{2}\right)\right) + \frac{2\sqrt{2}}{c} \sinh\left(\frac{c}{2} \left(\frac{\mathfrak{s} - s_0}{2}\right)\right) \right] \\ &\leq \sqrt{\frac{2 \max\{g_0(s_0), g_0(\mathfrak{s})\}}{\cosh\left(c \frac{\mathfrak{s}-s_0}{2}\right)}} \left[\frac{4\sqrt{2}}{c} \sinh\left(\frac{c}{4} (\mathfrak{s} - s_0)\right) \right]. \end{aligned}$$

Here the first inequality follows from the fact that

$$\sqrt{\cosh u} < \sqrt{2} \cosh\left(\frac{u}{2}\right).$$

For the second integral, if $\mathfrak{s} - s_0 \geq 1/c$, we have

$$\begin{aligned} &\sqrt{\frac{2c_2}{c_1^2 - c^2} \frac{1}{\sinh(c(\mathfrak{s} - s_0))}} \int_{s_0}^s \sqrt{\sinh(c(\mathfrak{s} - \mathfrak{x}))} d\mathfrak{x} \\ &\leq \sqrt{\frac{2c_2}{c_1^2 - c^2} \frac{1}{\sinh(c(\mathfrak{s} - s_0))}} \int_{s_0}^s \sqrt{\frac{e^{c(\mathfrak{s}-\mathfrak{x})}}{2}} d\mathfrak{x} \\ &\leq \frac{8}{c} \cdot \sqrt{\frac{c_2}{c_1^2 - c^2}}; \end{aligned}$$

if $\mathfrak{s} - s_0 < 1/c$, we have

$$\begin{aligned}
& \sqrt{\frac{2c_2}{c_1^2 - c^2} \frac{1}{\sinh(c(\mathfrak{s} - s_0))}} \int_{s_0}^s \sqrt{\sinh(c(\mathfrak{s} - \mathfrak{x}))} d\mathfrak{x} \\
&= \sqrt{\frac{2c_2}{c_1^2 - c^2}} \int_{s_0}^s \sqrt{\frac{\sinh(c(\mathfrak{s} - \mathfrak{x}))}{\sinh(c(\mathfrak{s} - s_0))}} d\mathfrak{x} \\
&\leq \frac{1}{c} \sqrt{\frac{2c_2}{c_1^2 - c^2}}.
\end{aligned}$$

Putting these together we proved this lemma. \square

Remark 4. By requiring that \mathfrak{b} is sufficiently large, we can make c_2 sufficiently small.

Now let's estimate the derivatives of z .

Lemma 13. *There exist $\delta > 0$, $\mathfrak{b} > 0$ and $\bar{\kappa} > 0$ such that, if*

$$\begin{aligned}
\sup_{(s,t) \in \theta} |\partial^\beta z_{out}(s,t)| &\leq \delta \\
a(s_0, 0) &\geq \mathfrak{b}
\end{aligned}$$

for multi-indices β with $|\beta| \leq l$, and

$$\begin{aligned}
\sup_{(s,t) \in \theta} |\partial^\beta (a(s,t) - Ts)| &\leq \delta \\
\sup_{(s,t) \in \theta} |\partial^\beta (\vartheta(s,t) - t)| &\leq \delta \\
\sup_{(s,t) \in \theta} |\partial^\beta z_{in}(s,t)| &\leq \delta
\end{aligned}$$

for those multi-indices β with $0 < |\beta| \leq l$, then for $s \in [s_0, \mathfrak{s}]$, we have

$$\begin{aligned}
\|\partial^\beta z(s)\|_0 &\leq C_\beta \max_{|\beta'| \leq |\beta|} \left\{ \|Q_0 \partial^{\beta'} z(s_0)\|_0, \|Q_0 \partial^{\beta'} z(\mathfrak{s})\|_0 \right\} \sqrt{\frac{\cosh(c_1(s - \frac{s_0 + \mathfrak{s}}{2}))}{\cosh(c_1(\frac{s_0 - \mathfrak{s}}{2}))}} \\
&+ C^\beta(c_2) \sqrt{\frac{\sinh(c(\mathfrak{s} - s))}{\sinh(c(\mathfrak{s} - s_0))}} + c_2 e^{-c_1(s - s_0)},
\end{aligned}$$

where

$$\mathfrak{s} := \sup \{s \in [s_0, s_1] \mid |\kappa_0(s')| \leq \bar{\kappa} \text{ for all } s' \in [s_0, s]\},$$

and $C_\beta, c_1 > 0$ are constants independent of s_0 and s_1 , $1 \leq |\beta| \leq l - 2$, and $C^\beta(c_2)$ is a function of c_2 independent of s_0 and s_1 , satisfying $\lim_{c_2 \rightarrow 0} C^\beta(c_2) = 0$, and l is the integer in Definition 1.

Proof. Let's prove the estimate for $|\beta| = 1$. The proof of estimates of higher derivatives is almost the same. Refer to Lemma A.6 in [5] for the estimates for all derivatives in the Cylindrical case.

Equation (52) can be rewritten as

$$z_s = A_0 z + \dot{\Delta} z_t + \ddot{\Delta} z_{out} + \ddot{\Delta} z - L, \quad (57)$$

with $\dot{\Delta} = \Delta_0 + \bar{\Delta}_0 \kappa_0$, $\ddot{\Delta} = \hat{\Delta}_0 + \bar{\Delta}_0 \kappa_0$, and $\ddot{\Delta} = [A(s) - \bar{A}(s)]$. Denote $\mathcal{W} := (Q_0 z, \frac{\partial}{\partial s}(Q_0 z), A_0 Q_0 z, \frac{\partial}{\partial s}(A_0 Q_0 z))$, then \mathcal{W} satisfies

$$\mathcal{W}_s = \mathcal{A}_0 \mathcal{W} + \mathcal{Q}_0 \dot{\Delta} \mathcal{W}_t + \mathcal{Q}_0 \ddot{\Delta} \mathcal{W}_{out} + \ddot{\Delta} \mathcal{W} - \mathcal{L},$$

where $\mathcal{A}_0 = \text{diag}(A_0, A_0, A_0, A_0)$, $\mathcal{Q}_0 = \text{diag}(Q_0, Q_0, Q_0, Q_0)$, and $\dot{\Delta}, \ddot{\Delta}, \ddot{\Delta}, \mathcal{L}$ satisfy similar estimates as $\dot{\Delta}, \ddot{\Delta}, \ddot{\Delta}, L$ respectively. Indeed, for $|\beta| = 1$ we could derive this equation by direct computation. For general β , we could derive this by induction on $|\beta|$. This equation is of the same type as the equation (57). Copying the proofs of Lemma 10, Lemma 11 and Lemma 12, we can get the desired estimate for \mathcal{W} . In particular, we get the estimates for $(Q_0 z)_s$ and $A_0 Q_0 z$.

From the equation $z_t = M_0 A_0 Q_0 z + M_0 Q_0 S_0 z_{out}$ we get the estimate for z_t . Applying P_0 to the equation (57), we get

$$(P_0 z)_s = P_0 \dot{\Delta} z_t + P_0 \ddot{\Delta} z_{out} + P_0 \ddot{\Delta} z - P_0 L.$$

This equation together with the estimate of $\ddot{\Delta} z$ (See formula 54) gives us the desired estimate for $P_0 z_s$. Then the estimate for z_s follows from $z_s = P_0 z_s + Q_0 z_s$. \square

Lemma 14. Let $\vartheta_0 = \int_0^1 [\vartheta(\frac{s_0+s}{2}, t) - Tt] dt$, $a_0 = \int_0^1 [a(\frac{s_0+s}{2}, t) - Ts_0] dt$, $\tilde{a} = a(s, t) - Ts - a_0$ and $\tilde{\vartheta} = \vartheta(s, t) - Tt - \vartheta_0$. Under the assumption of Lemma 13, we have for $s \in [s_0, \mathfrak{s}]$ and all multi index β with $|\beta| \leq l - 3$,

$$\begin{aligned} & \|\partial^\beta(\tilde{a}(s, t))\|^2, \left\| \partial^\beta(\tilde{\vartheta}(s, t)) \right\|^2 \\ & \leq C_1 \max_{|\beta'| \leq |\beta|+3} \{ \|Q_0 \partial^{\beta'} z(s_0)\|_0^2, \|Q_0 \partial^{\beta'} z(\mathfrak{s})\|_0^2 \} \\ & + C_1 \max \left\{ \|\tilde{a}(s_0, \cdot)\|^2 + \|\tilde{\vartheta}(s_0, \cdot)\|^2, \|\tilde{a}(\mathfrak{s}, \cdot)\|^2 + \|\tilde{\vartheta}(\mathfrak{s}, \cdot)\|^2 \right\} + o(c_2), \end{aligned}$$

where the norm $\|\cdot\|$ is L^2 -norm, $o(c_2)$ satisfies $\lim_{c_2 \rightarrow 0} o(c_2) = 0$, and C_1 is a constant independent of \tilde{u} .

Proof. We could modify the proofs of Lemmata 3.8-3.13 in [16] in the obvious way similar to what we did in the proof of Lemma 10 and then use Lemma 13 to prove this lemma. We omit the proof here, since essentially it is not new. \square

Remark 5. When \mathfrak{s} is infinity, we could get a better exponential decay estimate using the same proof, and in that case the term $o(c_2)$ can be replaced by $c_2 e^{-(s-s_0)}$.

Now we are ready to prove Theorem 1.

Proof. Let's follow the proof in [4]. By Theorem 3, we can find a sequence $s_{0m} \rightarrow \infty$ such that

$$\begin{aligned}\lim_{m \rightarrow \infty} u(s_{0m}, t) &= \gamma(Tt) \\ \lim_{m \rightarrow \infty} a(s_{0m}, t) &= \pm\infty\end{aligned}$$

for some T -periodic orbit γ of \mathbf{R}_∞ . From the proof of Theorem 3, we can further require for any multi-indices α with $|\alpha| > 0$ we have $\sup_{t \in S^1} \|\partial^\alpha z(s_{0m}, t)\| \rightarrow 0$ as $m \rightarrow +\infty$.

Given $\sigma > 0$ and let $\zeta_m > 0$ be the largest number such that $u(s, t) \in S^1 \times [-\sigma, \sigma]^{2n}$ for all $s \in [s_{0m}, s_{0m} + \zeta_m]$. Let $\theta_m := [s_{0m}, s_{0m} + \zeta_m] \times S^1$ and $\kappa_{0m}(s) := (\vartheta(s_{0m}, 0) - \vartheta(s, 0), z_{in}(s_{0m}, 0) - z_{in}(s, 0))$ and we can define the operator A_{0m} in the obvious way.

By Corollary 1, given $\delta > 0$ we have

$$\sup_{(s,t) \in \theta_m} |\partial^\beta (a(s, t) - Ts)| \leq \delta$$

for those multi-indices β with $0 < |\beta| \leq 3$, when m is large. This implies $a(s_{0m}, 0) \rightarrow +\infty$, as $m \rightarrow +\infty$. Notice that the other requirements in the Lemma 10 and Lemma 13 are also satisfied, i.e. given $\delta > 0$, there exists m_δ such that for $m > m_\delta$ we have

$$\sup_{(s,t) \in \theta_m} |\partial^\beta z_{out}(s, t)| \leq \delta$$

for multi-indices β with $|\beta| \leq l$, and

$$\sup_{(s,t) \in \theta_m} |\partial^\beta (\vartheta(s, t) - Tt)| \leq \delta \tag{58}$$

$$\sup_{(s,t) \in \theta_m} |\partial^\beta z_{in}(s, t)| \leq \delta$$

for those multi-indices β with $0 < |\beta| \leq l$. Indeed, if $\{(s_{m_k}, t_{m_k})\}$ violates one of these properties, we could define $\tilde{u}_{m_k}(s, t)$ to be $(a(s - s_{m_k}, t - t_{m_k}) - a(s_{m_k}, t_{m_k}), u(s - s_{m_k}, t - t_{m_k}))$. By Ascoli-Arzelà, we can extract a subsequence, still called $\tilde{u}_{m_k}(s, t)$, such that $\tilde{u}_{m_k}(s, t)$ converges in C_{loc}^∞ to a J_∞ -holomorphic cylinder \tilde{u}_∞ over a periodic orbit γ' of \mathbf{R}_∞ . Since \tilde{u}_∞ must satisfy those three properties, we get a contradiction.

By construction $|\langle z(s_{0m}), e \rangle_{0m}| \rightarrow 0$ and $\|Q_{0m} \partial^\alpha z(s_{0m})\| \rightarrow 0$, for all multi-index α with $|\alpha| \geq 0$. Let $\bar{\kappa}_m$ be the “ $\bar{\kappa}$ ” in Lemma 10 and Lemma 13 applied to $\tilde{u}|_{\theta_m}$ and let $\mathfrak{s}_m := \sup \{s \in [s_{0m}, s_{0m} + \zeta_m] \mid |\kappa_{0m}(s')| \leq \bar{\kappa}_m \text{ for all } s' \in [s_0, s]\}$,

and notice that actually $\bar{\kappa}_m$ can be chosen independent of m . We can extract a subsequence so that $u(\mathfrak{s}_m, t)$ converges to a closed Reeb orbit γ'' . Therefore, $\|Q_{0m}\partial^\alpha z(\mathfrak{s}_m)\| \rightarrow 0$, for all multi-indices α with $|\alpha| \geq 0$. $\langle z(\mathfrak{s}_m), e \rangle_0 \rightarrow 0$ and $\sup_{t \in S^1} \left| \frac{\partial}{\partial t} z_{in}(\mathfrak{s}_m, t) \right| \rightarrow 0$ give us $\sup_{t \in S^1} |z_{in}(\mathfrak{s}_m, t)| \rightarrow 0$. By Lemma 10 and Lemma 13, we have

$$\sup_{s \in [s_{0m}, \mathfrak{s}_m]} \|\partial^\beta z(s)\|_{0m} \rightarrow 0 \quad (59)$$

for $|\beta| \leq k$. Therefore,

$$\begin{aligned} & \sup_{(s,t) \in [s_{0m}, \mathfrak{s}_m] \times S^1} |z_{in}(s, t)| \\ & \leq \sup_{s \in [s_{0m}, \mathfrak{s}_m]} \|z_{in}(s, \cdot)\|_{C^0(S^1)} \\ & \leq \mathfrak{C} \sup_{s \in [s_{0m}, \mathfrak{s}_m]} \|z_{in}(s, \cdot)\|_{W^{1,2}(S^1)} \\ & \leq \mathfrak{C}_1 \left\{ \sup_{s \in [s_{0m}, \mathfrak{s}_m]} \left\| \frac{\partial}{\partial t} z_{in}(s, \cdot) \right\|_{0l} + \sup_{s \in [s_{0m}, \mathfrak{s}_m]} \|z_{in}(s, \cdot)\|_{0m} \right\} \\ & \rightarrow 0. \end{aligned}$$

Lemma 14 and formula (58) imply $|\vartheta(\mathfrak{s}_m, 0) - \vartheta(s_{0m}, 0)| \rightarrow 0$, as $m \rightarrow \infty$. Thus, we have $\mathfrak{s}_m = s_{0m} + \zeta_m$ for m large enough, and

$$\sup_{(s,t) \in [s_{0m}, s_{0m} + \zeta_m] \times S^1} |z(s, t)| \rightarrow 0$$

as $m \rightarrow \infty$. Therefore, $\zeta_m = +\infty$ for m large. \square

Furthermore, we can show the convergence of J -holomorphic curve is exponentially fast. Let's prove Theorem 2.

Proof. Now with the help of the previous lemmata, the proof of the third inequality is almost evident. Indeed, since $\mathfrak{s} = +\infty$, Lemma 11 becomes $g_0(s) \leq \left(g_0(s_0) + \frac{c_2}{c_1^2 - c^2} \right) e^{-c(s-s_0)}$. Consequently, in the proof Lemma 12, we could get

$$|\langle z(s), e \rangle_0| \leq \int_s^{+\infty} \left[2C \|Q_0 z(\mathfrak{x})\|_0 + c_2 e^{-c_1(\mathfrak{x}-s_0)} \right] d\mathfrak{x} \leq C' e^{-c(s-s_0)},$$

where C' is independent of s . Similarly, we could get the corresponding statement of Lemma 13 for $\mathfrak{s} = +\infty$.

The proof for the rest is a straightforward modification of the original proof in [15]. \square

So far we studied the behaviors of a finite energy J -holomorphic curve whose domain is an infinite cylinder. In order to compactify the moduli space of

holomorphic curves, we also need to understand the behavior of a finite energy J -holomorphic curve whose domain is a long but finite interval and whose ω energy is small. To do that, we first need the following Bubbling Lemma.

Lemma 15. (*Bubbling Lemma [5]*) *Let $(W = \mathbb{R} \times V, J_0)$ be a cylindrical almost complex manifold. There exists a constant $\hbar > 0$ depending only on (W, J_0, ω_0) where ω_0 is the 2-form in Definition 1 and 2, so that the following holds true. Let (J_n, ω_n) be a sequence of pairs satisfying (AC1)-(AC7) on W and converging to (J_0, ω_0) in C_{loc}^1 , i.e. for any compact subset $K \subset W$, both*

$$\|(J_n - J_0)|_K\|_{C^1} = \sup_{w \in K} (|\nabla(J_n - J_0)(w)| + |(J_n - J_0)(w)|)$$

and

$$\|(\omega_n - \omega_0)|_K\|_{C^1} = \sup_{w \in K} (|\nabla(\omega_n - \omega_0)(w)| + |(\omega_n - \omega_0)(w)|)$$

converge to 0. Consider a sequence of J_n -holomorphic maps $\tilde{u}_n = (a_n, u_n)$ from the unit disc $B(0, 1)$ to W satisfying $E_n(\tilde{u}_n) \leq C$ for some constant C , such that the sequence $a_n(0)$ are bounded, and such that $\|\nabla \tilde{u}_n(0)\| \rightarrow +\infty$ as $n \rightarrow +\infty$. Then there exist a sequence of points $z_n \in B(0, 1)$ converging to 0, sequences of positive numbers ε_n and R_n satisfying

$$\begin{aligned} \varepsilon_n &\rightarrow 0, & R_n &\rightarrow +\infty \\ \varepsilon_n R_n &\rightarrow +\infty, & |z_n| + \varepsilon_n &< 1 \end{aligned}$$

such that the rescaled maps

$$\begin{aligned} \tilde{u}_n^0 &: B(0, \varepsilon_n R_n) \rightarrow W \\ z &\mapsto \tilde{u}_n(z_n + R_n^{-1}z) \end{aligned}$$

converge in C_{loc}^∞ to a J_0 -holomorphic map $\tilde{u}^0 : \mathbb{C} \rightarrow W$ which satisfies $E_0(\tilde{u}^0) \leq C$ and $E_{\omega_0}(\tilde{u}^0) > \hbar$.

Moreover, this map is either a holomorphic sphere or a holomorphic plane asymptotic as $|z| \rightarrow \infty$ to a periodic orbit of the vector field \mathbf{R}_0 defined by $\mathbf{R}_0 = J_0 \left(\frac{\partial}{\partial r} \right)$.

Proof. The proof is almost the same as the proof in [5]. \square

The following theorem shows a property of a sufficiently long cylinder having small ω -area. It is needed in order to prove the compactness results for the moduli space of J -holomorphic curves in the Symplectic Field Theory. Refer to [16, 5] for the cylindrical case.

Theorem 4. *Suppose that J is an asymptotically almost complex structure on $W = \mathbb{R} \times V$. Given $E_0 > 0$ and $\varepsilon > 0$, there exist constants $\sigma, c > 0$ such that for every $R > c$ and every J -holomorphic cylinder $\tilde{u} = (a, u) : [-R, R] \times S^1 \rightarrow W$ satisfying the inequalities $E_\omega(\tilde{u}) < \sigma$ and $E(\tilde{u}) < E_0$, we have $u(s, t) \in B_\varepsilon(u(0, t))$ for all $s \in [-R + c, R - c]$ and all $t \in S^1$.*

Proof. The proof follows the scheme in [5] with some modification.

By contradiction, assume that there exist sequences $c_n \rightarrow +\infty$, $R_n > c_n$ and $\tilde{u}_n = (a_n, u_n) : [-R_n, R_n] \times S^1 \rightarrow \mathbb{R} \times M$. \tilde{u}_n is J -holomorphic satisfying $E(\tilde{u}_n) \leq E_0$, $E_\omega(\tilde{u}_n) \rightarrow 0$, and $u_n(s_n, t_n) \notin B(u_n(0, t_n), \epsilon)$ for some $s_n \in [-k_n, k_n]$, $k_n = R_n - c_n$ and $t_n \in S^1$. By the proof of Proposition 2 together with the Bubbling Lemma, $\|\nabla \tilde{u}_n\|$ is uniformly bounded on each compact subset. We could extract a subsequence of n , still denoted by n , such that $a_n(s_n, t_n) \rightarrow +\infty$. This is because otherwise, we could get a contradiction as in the proof of Proposition 2. Now define $\tilde{u}_n^0(s, t) := (a_n^0, u_n^0) = (a_n(s, t) - a_n(s_n, t_n), u_n(s, t))$. Hence, by Ascoli-Arzelà, we can extract a subsequence still called \tilde{u}_n^0 converging to a J_∞ -holomorphic cylinder $\tilde{u} : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times M$. \tilde{u} satisfies $E_\omega(\tilde{u}) = 0$ and $E(\tilde{u}) \leq E_0$, so \tilde{u} is a trivial vertical cylinder over some periodic orbit γ . Let's choose a neighborhood around γ and pick the coordinate as in Lemma 3, and show that

$$\sup_{(s,t) \in [-k_n, k_n] \times S^1} |\partial^\beta z_{out,n}(s, t)| \rightarrow 0 \quad (60)$$

for multi-indices β with $|\beta| \leq 3$ and

$$\sup_{(s,t) \in [-k_n, k_n] \times S^1} |\partial^\beta (a_n(s, t) - Ts)| \rightarrow 0 \quad (61)$$

$$\sup_{(s,t) \in [-k_n, k_n] \times S^1} |\partial^\beta z_{in,n}(s, t)| \rightarrow 0 \quad (62)$$

$$\sup_{(s,t) \in [-k_n, k_n] \times S^1} |\partial^\beta (\vartheta_n(s, t) - Tt)| \rightarrow 0 \quad (63)$$

for multi-indices β with $0 < |\beta| \leq 3$, when $n \rightarrow +\infty$.

If this were not true, suppose there exists a subsequence of $\{n\}$ still denoted by $\{n\}$ such that (s'_n, t'_n) violates one of these properties. Then we could do the same argument using (s'_n, t'_n) instead of (s_n, t_n) as above, and get a vertical trivial cylinder contradicting to the fact that (s'_n, t'_n) violates one of these properties.

Define A_{0n} and Q_{0n} in the obvious way using γ and $s_{0n} = 0$. Then we could apply Lemma 10, Lemma 11, Lemma 12, and Lemma 13 to each $\tilde{u}_n|_{[-k_n, k_n]}$, and get $\sup_{s \in [-k_n, k_n]} \|Q_{0n} z_n(s)\|_{0,n} \rightarrow 0$. Then the Sobolev embedding theorem tells us $\kappa_{0n} \rightarrow 0$ as $n \rightarrow +\infty$. This contradicts to the assumption that $u_n(s_n, t_n) \notin B(u_n(0, t), \epsilon)$. \square

We need the following theorem later to prove the surjectivity of the gluing map. After we proved all the previous lemmata and theorems, the proof of the following theorem is standard.

Theorem 5. *Suppose that J is an asymptotically almost complex structure on $W = \mathbb{R} \times V$. Given $E_0 > 0$ and $\varepsilon > 0$, there exist constants $\sigma, c, \flat, \nu > 0$ such that for every $R > c$ and every J -holomorphic cylinder $\tilde{u} = (a, u) : [-R, R] \times S^1 \rightarrow \mathbb{R} \times V$ satisfying the inequalities $a > \flat$, $E_\omega(\tilde{u}) < \sigma$ and $E(\tilde{u}) < E_0$, there exist $\gamma \in \mathcal{P}$ and a coordinate around γ as in Lemma 3 such that we have*

$$\begin{aligned}
|D^\beta\{a(s, t) - Ts - a_0\}| &\leq M_\beta \frac{\cosh(2\nu s)}{\cosh(2\nu(R-c))} + C_\beta e^{-c_\beta(s+R-c)}, \\
|D^\beta\{\vartheta(s, t) - Tt - \vartheta_0\}| &\leq M_\beta \frac{\cosh(2\nu s)}{\cosh(2\nu(R-c))} + C_\beta e^{-c_\beta(s+R-c)}, \\
|D^\beta z(s, t)| &\leq M_\beta \frac{\cosh(2\nu s)}{\cosh(2\nu(R-c))} + C_\beta e^{-c_\beta(s+R-c)},
\end{aligned}$$

for $s \in [-R+c, R-c]$, $t \in S^1$, and $\beta \in \mathbb{N} \times \mathbb{N}$ such that $|\beta| \leq l-3$, where $M_\beta, C_\beta, c_\beta$ are constants independent of \tilde{u} , C_β converges to 0 as \mathfrak{b} converges to $+\infty$, and M_β, c_β are independent of \mathfrak{b} .

4 Almost complex manifolds with asymptotically cylindrical ends

In this section, we introduce the notion of almost complex manifolds with asymptotically cylindrical ends. Obviously the results about the behaviors of J -holomorphic curves and the proofs of the results are almost the same as in the asymptotically cylindrical case, so please refer to previous sections.

4.1 Definition

Let (W, J) be a $2n+2$ dimensional noncompact almost complex manifold, and E_\pm be an open subset containing the positive (negative) end of W . Assume that E_\pm is diffeomorphic to $\mathbb{R}^\pm \times V_\pm$, where V_\pm is a $2n+1$ dimensional closed manifold. Assume that there exist a J compatible symplectic form ω' on W , and $(\mathbb{R}^\pm \times V_\pm, J)$ is asymptotically cylindrical at positive (negative) infinity, then we say (W, J) is an almost complex manifold with asymptotically cylindrical positive (negative) end.

Example 3. [5] Let (X, ω', J) be an almost Kähler manifold, and $Y \subset X$ is an embedded closed almost Kähler submanifold. We claim that $(X \setminus Y, J|_{X \setminus Y})$ has asymptotically cylindrical negative end. Let N be the normal bundle of Y in X with the metric $\omega'(\cdot, J\cdot)|_Y$, V be the associated unit sphere bundle of N defined by $V := \{(u, y) \in N \mid |u| = 1\}$, and U_ϵ be the disc bundle over Y defined by $U_\epsilon := \{(u, y) \in N \mid |u| \leq \epsilon\}$. For small enough $\epsilon > 0$, U_ϵ is diffeomorphic to a tubular neighborhood of Y in X via the exponential map with respect to the metric $\omega'(\cdot, J\cdot)$. Since U_ϵ is also diffeomorphic to $(-\infty, \log \epsilon] \times V$ via the map $(u, y) \mapsto (\log |u|, u/|u|, y)$, we get an almost complex structure and a 2-form on $(-\infty, \log \epsilon] \times V$, still denoted by J and ω' respectively. Let (s, v) be the coordinate on $(-\infty, \log \epsilon] \times V$, and define $\mathbf{R} = J(\frac{\partial}{\partial s})$, $\xi = TV \cap JTV$. Then we have $T((-\infty, \log \epsilon] \times V) = \mathbb{R}(\frac{\partial}{\partial s}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$ and denote π_ξ to be the projection onto ξ . Also for any $w \in T_{(s,v)}[(-\infty, \log \epsilon] \times V] \cong U_\epsilon$, we can write w as $w = w_u + w_y$ with w_u in the fiber direction and w_y tangent to the

base Y using the metric $\omega'(\cdot, J\cdot)|_Y$. Define a 2-form ω on $(-\infty, \log \epsilon] \times V$ by $\omega(v, w) = \omega'(\pi_\xi(e^{-s}v_u + v_y), \pi_\xi(e^{-s}w_u + w_y))$ for $v, w \in T_{(s,r)} [(-\infty, \log \epsilon] \times V]$. It is easy to check that the pair (ω, J) satisfies (AC1)-(AC7).

In particular, if we pick Y to be a point in X , we get Example 2 as a special case.

4.2 Energy of J -holomorphic curves

Let w be a J -holomorphic map from (S, j) to (W, J) , and define

$$E_\omega(w) = \int_{w^{-1}(W-E_+-E_-)} w^* \omega' + \int_{w^{-1}(E_+)} w^* \omega + \int_{w^{-1}(E_-)} w^* \omega.$$

$$E_\lambda(w) = \sup_{\phi \in \mathcal{C}_+} \int_{w^{-1}(E_+)} w^*(\phi \sigma \wedge \lambda) + \sup_{\phi \in \mathcal{C}_-} \int_{w^{-1}(E_-)} w^*(\phi \sigma \wedge \lambda),$$

where

$$\mathcal{C}_+ = \{\phi \in C_c^\infty(\mathbb{R}^+, [0, 1]) \mid \int \phi = 1\}$$

$$\mathcal{C}_- = \{\phi \in C_c^\infty(\mathbb{R}^-, [0, 1]) \mid \int \phi = 1\},$$

and

$$E(w) = E_\omega(w) + E_\lambda(w).$$

Now based on the analytical results in previous sections, we can get the following compactness results.

Theorem 6. *The moduli spaces of proper stable holomorphic buildings with bounded Hofer's energy, whose domains have a fixed number arithmetic genus and a fixed number of marked points, are compact.*

See 8.1 and 8.2 in [5] for the definition of moduli space of stable holomorphic buildings in manifolds with cylindrical ends and the topology of the moduli space of holomorphic buildings; see 4.1-4.5 in [5] for other related definitions. Readers could refer to Appendix 6.1 and 6.2 of this paper for the case with presence of Lagrangian boundary conditions.

Based on the results in the previous parts of this paper, the proof of this theorem is a minor modification of the proof as in 10.2 in [5] in the obvious way, so we omit it. For a special case in Section 5, we give a proof in Theorem 10, which is slightly different from the situation here because we have to deal with the Lagrangian boundary condition in Section 5.

5 An application to Lagrangian Intersection Theory

In this section, we give an application of the results established in the previous sections to Lagrangian surgery. In particular, we give another proof of Theorem Z in [9, 10, 11]. Theorem Z is the main theorem in [11]. It relates the moduli space of J -holomorphic triangles w_{tri} with boundaries lying in three transversal Lagrangian submanifolds L_0, L_1 and L_2 to the moduli space of nearby J -holomorphic strips with boundaries lying in L_0 and the Lagrangian connected sum $L_1 \sharp_\epsilon L_2$. Roughly speaking, it says the second moduli space is a fiber bundle over the first moduli space with fiber diffeomorphic to a point or S^{n-2} depending on the sign of ϵ , where n is the dimension of the Lagrangian submanifolds. The relation between Theorem Z and Mirror Symmetry can be found in [8, 11].

The advantages of the new proof are: first, we don't need to assume the almost complex structure J is integrable near $L_1 \cap L_2$; second, the new proof uses the notion of holomorphic buildings in Symplectic Field Theory which is more intuitive than the mere estimates as in the original proofs; finally, the new proof is given in a more systematic way.

For completeness sake, we will recall some definitions and results from [11]. Readers could refer to [11] for some details.

5.1 Lagrangian surgery

Let (M, ω', J) be an almost Kähler manifold, and L_1 and L_2 be two Lagrangian submanifolds transversally intersecting at the point p_{12} . We can pick an open neighborhood B around p_{12} symplectomorphic to an open neighborhood $B(\varepsilon_0)$ of $0 \in \mathbb{C}^n$ with the standard symplectic form $\Sigma dx_i \wedge dy_i$, such that L_1 and L_2 locally look like \mathbb{R}^n and

$$\left\{ (e^{\alpha_1 \sqrt{-1}} v_1, \dots, e^{\alpha_2 \sqrt{-1}} v_n) \mid v_1, \dots, v_n \in \mathbb{R}^n \right\}$$

respectively, and the almost complex structure J restricted to the origin is the same as the standard complex structure of \mathbb{C}^n restricted to the origin. Here $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n < \pi$ are called Kähler angles. In this paper, let's restrict ourselves to the case $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$. Also to simplify notation, we can assume $\varepsilon_0 = 1$, which is not essential.

Let's define a new Lagrangian submanifold $H_{\epsilon_1} := \gamma_{\epsilon_1} \cdot S_{\mathbb{R}^n}^{n-1} \subset \mathbb{C}^n$, where $S_{\mathbb{R}^n}^{n-1}$ is the unit sphere in $\mathbb{R}^n \subset \mathbb{C}^n$, and

$$\gamma_{\epsilon_1} = \left\{ r e^{\sqrt{-1}\theta} \in \mathbb{C} \mid |2\epsilon_1|^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi\theta}{\alpha}\right), \theta \in (0, \alpha) \right\} \text{ if } \epsilon_1 > 0,$$

$$\gamma_{\epsilon_1} = \left\{ r e^{\sqrt{-1}\theta} \in \mathbb{C} \mid |2\epsilon_1|^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi(\theta - \alpha)}{\pi - \alpha}\right), \theta \in (0, \alpha) \right\} \text{ if } \epsilon_1 < 0.$$

Let's modify H_{ϵ_1} to get a Lagrangian submanifold $(H_{\epsilon_1}^\alpha)'$ which agrees with $\mathbb{R}^n \cup e^{\alpha\sqrt{-1}}\mathbb{R}^n$ outside $B(0, 2S_0\sqrt{|\epsilon_1|})$, for some $S_0 > 0$. Let's focus on $\epsilon_1 > 0$ case ($\epsilon_1 < 0$ case is similar). Consider a function $\theta(r) : [\sqrt{2|\epsilon|}, +\infty) \rightarrow [0, \alpha/2]$ satisfying $\theta(r) = 0$ for $r \geq 2S_0\sqrt{|\epsilon_1|}$; for $r \leq S_0\sqrt{|\epsilon_1|}$, θ is defined by $|2\epsilon|^{\frac{\pi}{2\alpha}} = r^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi\theta(r)}{\alpha}\right)$ and $\frac{d\theta}{dr} \leq 0$. Then we put

$$(\gamma_{\epsilon_1}^\alpha)' = \left\{ re^{\sqrt{-1}\theta(r)} \mid r \in [\sqrt{2|\epsilon_1|}, \infty) \right\} \cup \left\{ re^{\sqrt{-1}(\alpha-\theta(r))} \mid r \in [\sqrt{2|\epsilon_1|}, \infty) \right\}$$

and define $(H_{\epsilon_1}^\alpha)' = (\gamma_{\epsilon_1}^\alpha)' \cdot S_{\mathbb{R}^n}^{n-1} \subset \mathbb{C}^n$. By identifying the local model with the neighborhood B we replace L_1 and L_2 by $L_{\epsilon_1} = \{(H_{\epsilon_1}^\alpha)' \cap B\} \cup \{L_1 \cup L_2 - B\}$. For the case $\epsilon_1 > 0$, we call this process a positive surgery; for the case $\epsilon_1 < 0$, we call it a negative surgery.

5.2 Holomorphic triangles

Let L_0, L_1 and L_2 be a triple of Lagrangian submanifolds of (M, ω', J) , which are mutually transversal. Assume the Kähler angle between L_1 and L_2 all equal to α . Consider a J -holomorphic triangle w_{tri} from the unit disk $D^2 \subset \mathbb{C}$ to M , such that

$$\begin{cases} w_{tri}(-1) = p_{01}, & w_{tri}(\overrightarrow{(-1, -\sqrt{-1})}) \subset L_1, \\ w_{tri}(1) = p_{02}, & w_{tri}(\overrightarrow{(-\sqrt{-1}, 1)}) \subset L_2, \\ w_{tri}(-\sqrt{-1}) = p_{12}, & w_{tri}(\overrightarrow{(1, -1)}) \subset L_0, \\ w_{tri} \text{ is Fredholm regular,} \\ \text{The multiplicity of } w_{tri} \text{ at } -\sqrt{-1} \text{ is one, (see Remark 6)} \end{cases} \quad (64)$$

where we use the notation \overrightarrow{ab} to denote the arc on the unit circle from point a to point b using the standard boundary orientation, and $p_{ij} \in L_i \cap L_j$, and Fredholm regular means that $D\bar{\partial}$ is surjective.

Remark 6. If we identify $\mathbb{R} \times [0, 1]$ with the domain $D^2 \setminus \{-\sqrt{-1}, \sqrt{-1}\} \subset \mathbb{C}$ via the map χ ,

$$(\tau, t) \mapsto \frac{\sqrt{-1}e^{\pi(\tau+\sqrt{-1}t)} + 1}{e^{\pi(\tau+\sqrt{-1}t)} + \sqrt{-1}}$$

and identify $\mathbb{C}^n \setminus \{0\}$ with $\mathbb{R} \times S^{2n-1}$ via the map φ defined by $z \mapsto (\log|z|, z/|z|)$, then we say the J -holomorphic triangle w_{tri} has multiplicity one if

$$\lim_{\tau \rightarrow -\infty} w_{tri}(\tau, t) = e^{\alpha\sqrt{-1}t}a$$

uniformly in t , for some $a \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$. Notice that $x(t) = e^{\alpha\sqrt{-1}t}a$ ($0 \leq t \leq 1$) is a minimal solution of $\dot{x} = \mathbf{R}_{-\infty}(x)$ with two ends lying in \mathbb{R}^n and $e^{\alpha\sqrt{-1}}\mathbb{R}^n$ respectively, i.e. a simple Reeb chord. If the almost complex structure

J is integrable near 0, we know $w(\tau, t)$ has to converge to a multiple Reeb chord by Section 3.4 in [11], and if we assume the “multiplicity” is one, we get $w(\tau, t)$ converges to $e^{\alpha\sqrt{-1}t}a$ for some $a \in S^{2n-1}$. If we have some sort of Carleman Similarity Principle for the case with two transversal Lagrangian submanifolds, we could easily see that the J -holomorphic curve converges to some solution of $\dot{x} = \mathbf{R}_{-\infty}(x)$. However, at this moment we don’t know whether this is always the case, and it’s a work in progress. It is easy to see that converging to the Reeb chord is equivalent to having finite Hofer’s energy in this setting, so equivalently we could assume that the Hofer’s energy is so small that the J -holomorphic curve has to converge to some simple solution of $\dot{x} = \mathbf{R}_{-\infty}(x)$.

Let’s denote the moduli space of w_{tri} satisfying (64) by $\mathcal{M}((L_0, L_1, L_2), J)$. We then perform Lagrangian surgery at $p_{12} \in L_1 \cap L_2$ and get L_{ϵ_1} . Consider the set of J -holomorphic 2-gons $w : D^2 \rightarrow M$ satisfying

$$\begin{cases} w(\overrightarrow{1, -1}) \subset L_0 & w(\overrightarrow{-1, 1}) \subset L_{\epsilon_1} \\ w(-1) = p_{01} & w(1) = p_{12}. \end{cases} \quad (65)$$

Let’s denote the set of w ’s satisfying (65) by $\widetilde{\mathcal{M}}((L_{\epsilon_1}, L_0), J)$ and its quotient under the action of $Aut(D^2, (-1, 1))$ by $\mathcal{M}((L_{\epsilon_1}, L_0), J)$. Let’s denote by

$$\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$$

the subset of $\mathcal{M}((L_{\epsilon_1}, L_0), J)$ consisting of $[w]$ satisfying

$$\sup_{z \in D^2} dist_{g_M}(w(z), w_{tri}(z)) \leq \epsilon_2,$$

where g_M is the Riemannian metric on M given by $g_M(\cdot, \cdot) = \omega'(\cdot, J\cdot)$.

The following theorem is a generalization of the main theorem in [11], which is also the main theorem for this section. The proof of this theorem will be given step by step.

Theorem 7. *Assume that $w_{tri} \in \mathcal{M}((L_0, L_1, L_2), J)$ is isolated. Then for each sufficiently small ϵ_2 and ϵ_1 with $|\epsilon_1| < \epsilon_2^{100}$ we have the following*

If $\epsilon_1 < 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ consists of one point which is Fredholm regular;

If $\epsilon_1 > 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ is diffeomorphic to S^{n-2} . Each element of it is Fredholm regular.

Remark 7. In [11], the above theorem is proven under an additional requirement that J is integrable near p_{12} . The condition of w_{tri} being isolated can be weakened (see [11]), and the proof is similar.

We will use the method of gluing to prove this theorem. To do this, we need to study

5.3 Local model of holomorphic discs in \mathbb{C}^n .

Consider the Lagrangian subspaces \mathbb{R}^n and $e^{\sqrt{-1}\alpha}\mathbb{R}^n$ in \mathbb{C}^n , where $0 < \alpha < \pi$. Let's identify the $\mathbb{C}^n \setminus \{0\}$ with $\mathbb{R} \times S^{2n-1}$ via the map $z \mapsto (\log |z|, z/|z|)$. Fix $a \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$ and consider all holomorphic maps $w : \mathbb{H} \rightarrow \mathbb{C}^n$ satisfying

$$\begin{cases} w(\partial\mathbb{H}) \subset (H_{\epsilon_1}^\alpha)', \\ \text{There exist } c, C > 0 \text{ and } \tau_0 \in \mathbb{R} \text{ such that} \\ e^{-\alpha\tau} \left| w(e^{\pi(\tau+\sqrt{-1}t)}) - e^{\alpha(\tau-\tau_0+\sqrt{-1}t)}a \right| \leq Ce^{-c\tau} \text{ for all } (\tau, t) \in \mathbb{R}^+ \times [0, 1]. \end{cases} \quad (66)$$

We denote the set of all such w 's by $\widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H_{\epsilon_1}^\alpha)', a)$, and denote

$$\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{\epsilon_1}^\alpha)', a) = \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H_{\epsilon_1}^\alpha)', a) / \text{Aut}\{\mathbb{H}\}.$$

Theorem 8. *There exists a constant $S_0(\alpha)$ independent of ϵ_1 , such that for all $S_0 > S_0(\alpha)$ we have*

If $\epsilon > 0$, $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{\epsilon_1}^\alpha)', a)$ has only one element;

If $\epsilon < 0$, $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{\epsilon_1}^\alpha)', a)$ is diffeomorphic to S^{n-2} . Moreover, in either case, elements in $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{\epsilon_1}^\alpha)', a)$ are regular.

Proof. Refer to 6.3 in [11]. □

We denote by $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{\pm 1}^\alpha)', a)$ the set of all $w_{lmd} \in \widetilde{\mathcal{M}}(\mathbb{H}, \mathbb{C}^n; (H_{\pm 1}^\alpha)', a)$ satisfying

$$\text{Ref}_{\alpha/2}(w_{lmd}(z)) = w_{lmd}(-\bar{z}),$$

and

$$e^{-\alpha\tau} \left| w_{lmd}(e^{\pi(\tau+\sqrt{-1}t)}) - e^{\alpha(\tau+\sqrt{-1}t)}a \right|_{\mathbb{C}^n} \leq Ce^{-c\tau},$$

for $\tau \geq 0$, and some constant C independent of w_{lmd} , where $\text{Ref}_{\alpha/2}$ is the reflection along $e^{\sqrt{-1}\frac{\alpha}{2}}\mathbb{R}^n$ in \mathbb{C}^n .

Lemma 16. *For each $[w] \in \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{\pm 1}^\alpha)', a)$, there exists exactly one element $w_{lmd} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{\pm 1}^\alpha)', a)$ such that $[w_{lmd}] = [w]$.*

Proof. Refer to Lemma 7.3 in [11]. □

From now on, **let's assume the surgery parameter ϵ_1 is negative**, because the proofs in both cases are similar and the proof in ϵ_1 negative case is harder than ϵ_1 positive case.

5.4 Compactness

We will construct a gluing map to prove Theorem 7. However, let's study the “inverse” of gluing first, which is slightly easier, i.e. we will study the compactification of the moduli space of holomorphic strips first.

Consider arbitrary sequences of $\epsilon_{1,i}, \epsilon_{2,i} > 0$ and J -holomorphic maps $w_i : D^2 \rightarrow M$ such that

$$\begin{cases} \lim_{i \rightarrow \infty} \epsilon_{1,i} = 0 = \lim_{i \rightarrow \infty} \epsilon_{2,i}, & w_i(-1) = p_{01}, w_i(1) = p_{20}, \\ w_i(\overrightarrow{-1, 1}) \in L_{-\epsilon_{1,i}}, & w_i(\overrightarrow{1, -1}) \in L_0, \\ \text{dist}_{g_M}(w_{tri}(z), w_i(z)) < \epsilon_{2,i}, & \epsilon_{1,i} < \epsilon_{2,i}^{100}. \end{cases} \quad (67)$$

where g_M is the Riemannian metric on M given by $g_M(\cdot, \cdot) = \omega'(\cdot, J\cdot)$.

We want to prove that there exists a subsequence of w_i converging to a holomorphic building of height 1|1 consisting of w_{tri} and w_{lmd} in the sense of Symplectic Field Theory. Refer to Appendix 6.2 for the definition of convergence to holomorphic buildings.

Uniform energy bound

To prove convergence, we need to get uniform energy bound. Let's recall the set up of 2.1. We pick an open neighborhood B around p_{12} symplectomorphic to an open neighborhood $B(\varepsilon_0)$ (We assume the radius $\varepsilon_0 = 1$ for the simplicity of notation) of 0 inside \mathbb{C}^n with the standard symplectic form $\omega_0 = \Sigma dx_i \wedge dy_i$, such that L_1 and L_2 locally look like \mathbb{R}^n and $e^{\alpha\sqrt{-1}}$, and $J(0) = i(0)$. Identify $B(1) \setminus \{0\}$ with $\mathbb{R}^- \times S^{2n-1}$ via $z \mapsto (\log |z|, \frac{z}{|z|})$. Let (r, Θ) be the coordinate of $\mathbb{R}^- \times S^{2n-1}$, and define $\xi := JTV \cap TV$, $\mathbf{R} := J(\frac{\partial}{\partial r})$, $\omega(u, v) := e^{-2r}\omega_0(\pi_\xi u, \pi_\xi v)$, and the 1-form λ by: $\lambda|_\xi = 0$, $\lambda(\frac{\partial}{\partial r}) = 0$, $\lambda(\mathbf{R}) = 1$. It's not hard to see that when restricted to S^{2n-1} , \mathbf{R}_∞ is the Reeb vector field of S^{2n-1} with the standard contact form λ_∞ .

Let $r_0 < 0$ to be determined, and pick $r_i \rightarrow -\infty$ as $i \rightarrow +\infty$. Denote the open subset $w_i^{-1}[(r_i, r_0) \times S^{2n-1}] \subset D^2$ by U_i , and then we have $\partial U_i = \partial_1 U_i - \partial_2 U_i - \cup_j \partial B_{ij}$, where B_{ij} 's are components of $w_i^{-1}([-\infty, r_i])$, $\partial_1 U_i$ is the pre-image of $\{r_0\} \times S^{2n-1}$, $\partial_2 U_i$ is to the pre-image of the Lagrangian submanifold $L_{-\epsilon_{1,i}}$, and the orientations are chosen so that this formula is true.

Lemma 17. $\|\nabla w_i\| < C < +\infty$, where the norm is computed with respect to the Euclidean metrics on $D^2 \subset \mathbb{C}$ and g_M on M , and the constant C is independent of i .

Proof. Suppose not, then there exists a bubble, which contradicts the requirement that $\text{dist}_{g_M}(w_{tri}(z), w_i(z)) < \epsilon_{2,i}$. \square

From Stokes theorem, we have

$$\int_{\partial_1 U_i} w_i^* \lambda_\infty - \int_{\partial_2 U_i} w_i^* \lambda_\infty - \Sigma b_{ij} = \int_{U_i} w_i^* d\lambda_\infty, \quad (68)$$

where $b_{ij} := \int_{\partial B_{ij}} w_i^* \lambda_\infty$.

Lemma 18. $b_{ij} \geq 0$.

Proof. Since the orientation of ∂B_{ij} comes from the orientation of B_{ij} , we get

$$\begin{aligned} \int_{\partial B_{ij}} w_i^* \lambda_{-\infty} &= e^{-r_i} \int_{\partial B_{ij}} w_i^* (e^r \lambda_{-\infty}) = e^{-r_i} \int_{B_{ij}} w_i^* d(e^r \lambda_{-\infty}) \\ &= e^{-r_i} \int_{B_{ij}} w_i^* \omega_0 \geq 0. \end{aligned} \quad (69)$$

□

Given $\delta > 0$, there exist r_0 and i_0 , such that for all $i \geq i_0$ we have the following:

Lemma 19.

$$\left| \int_{\partial_1 U_i} w_i^* \lambda_{-\infty} - \alpha \right| \leq \delta.$$

Proof. In the proof of this lemma there is no difference between cylindrical and asymptotically cylindrical case. This follows from Corollary 8.6 in [11]. □

For the second term in equation (68), we have

Lemma 20.

$$\left| \int_{\partial_2 U_i} w_i^* \lambda_{-\infty} - (\alpha - \pi) \right| \leq \delta.$$

Proof. Since $d\lambda_{-\infty}|_{TL_{-\epsilon_{1,i}}} = 0$, in order to compute $\int_{\partial_2 U_i} w_i^* \lambda_{-\infty}$, we can deform $w_i(\partial_2 U_i)$ inside $L_{-\epsilon_{1,i}}$ without changing the value of the integral. Then we can either use the computation in Lemma 8.21 in [11], or we can further show that $d\lambda_{-\infty}|_{\text{Span}_{\mathbb{C}}\{a\}} = 0$, where $a \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$, and deform $w_i(\partial_2 U_i)$ further to make the computation trivial. □

Proposition 4. $E_\omega(w_i) \leq \pi + 3\delta$.

Proof. We have

$$E_\omega(w_i) = \int_{U_i} w_i^* \omega = \int_{U_i} w_i^* (d\lambda_{-\infty}) + \int_{U_i} w_i^* (\omega - d\lambda_{-\infty}). \quad (70)$$

By Lemma 19 and Lemma 20, the first term becomes

$$\int_{U_i} w_i^* (d\lambda_{-\infty}) = \int_{\partial_1 U_i} w_i^* \lambda_{-\infty} - \int_{\partial_2 U_i} w_i^* \lambda_{-\infty} \leq \alpha - (\alpha - \pi) + 2\delta = \pi + 2\delta. \quad (71)$$

For the second term, by Lemma 17 we have

$$\int_{U_i} w_i^* (\omega - d\lambda_{-\infty}) \leq \int_{U_i} (\omega - d\lambda_{-\infty}) \left(\frac{\partial w_i}{\partial x}, \frac{\partial w_i}{\partial y} \right) dx \wedge dy \leq C^2 \cdot \text{Area}(U_i) \leq \delta. \quad (72)$$

□

Proposition 5. $E_\lambda(w_i) \leq \alpha + 6\delta$.

Proof. Given $\phi \in \mathcal{C}_- = \{\phi \in C_c^\infty(\mathbb{R}^-, [0, 1]) \mid \int \phi = 1\}$, let $\Phi(r) = \int_{-\infty}^r \phi(l)dl$, by a similar computation as in Lemma 20, we have

$$\int_{\partial_2 U_i} w_i^*(\Phi \lambda_{-\infty}) \geq \alpha - \pi - \delta.$$

While,

$$\begin{aligned} \int_{U_i} w_i^* d(\Phi \lambda_{-\infty}) &= \int_{\partial_1 U_i} w_i^*(\Phi \lambda_{-\infty}) - \int_{\partial_2 U_i} w_i^*(\Phi \lambda_{-\infty}) - \sum_j \int_{\partial B_{ij}} w_i^*(\Phi \lambda_{-\infty}) \\ &\leq \Phi(r_0) \int_{\partial_1 U_i} w_i^* \lambda_{-\infty} + (\pi - \alpha + \delta) - \Phi(r_i) \sum_j \int_{\partial B_{ij}} w_i^* \lambda_{-\infty} \\ &\leq 1 \cdot (\alpha + \delta) + (\pi - \alpha + \delta) - 1 \cdot 0 = \pi + 2\delta, \end{aligned} \quad (73)$$

and by (72)

$$\int_{U_i} w_i^*(\Phi d\lambda_{-\infty}) = \int_{U_i} w_i^*(\Phi \omega) + \int_{U_i} w_i^*(\Phi(d\lambda_{-\infty} - \omega)) \geq -\delta. \quad (74)$$

From (73) and (74), we get

$$\begin{aligned} \int_{U_i} w_i^*(\phi dr \wedge \lambda_{-\infty}) &= \int_{U_i} w_i^* d(\Phi \lambda_{-\infty}) - \int_{U_i} w_i^*(\Phi d\lambda_{-\infty}) \\ &\leq \pi + 2\delta + \delta \\ &\leq \pi + 3\delta. \end{aligned} \quad (75)$$

On the other hand,

$$\begin{aligned} \int_{U_i} w_i^*(\phi \sigma \wedge \lambda) &= \int_{U_i} w_i^*(\phi dr \wedge \lambda_{-\infty}) + \int_{U_i} w_i^*(\phi \sigma \wedge \lambda - \phi dr \wedge \lambda_{-\infty}) \\ &\leq \int_{U_i} w_i^*(\phi dr \wedge \lambda_{-\infty}) + \int_{U_i} w_i^* e^{2r} (\phi \omega + \phi \sigma \wedge \lambda) \\ &\leq \int_{U_i} w_i^*(\phi dr \wedge \lambda_{-\infty}) + e^{2r_0} \int_{U_i} w_i^* \omega + e^{2r_0} \int_{U_i} w_i^*(\phi \sigma \wedge \lambda). \end{aligned} \quad (76)$$

The third inequality follows from the fact $0 \leq \phi \leq 1$. Therefore, from (75), (76) and Proposition 4 we derive

$$\begin{aligned} E_\lambda(w_i) &= \sup_{\phi \in \mathcal{C}_-} \int_{U_i} w_i^*(\phi \sigma \wedge \lambda) \\ &\leq \frac{1}{1 - e^{2r_0}} \int_{U_i} w_i^*(\phi dr \wedge \lambda_{-\infty}) + \frac{e^{2r_0}}{1 - e^{2r_0}} (\pi + 1) \\ &\leq \pi + 4\delta. \end{aligned} \quad (77)$$

□

Altogether, we get

Proposition 6. $E(w_i) = E_\lambda(w_i) + E_\omega(w_i) \leq \pi + 7\delta$.

Theorem 9. w_i does not hit the origin.

Proof. For fixed i , the energy bound and Theorem 1 imply that w_i has to converge to some Reeb orbits near infinity inside B_{ij} , if the set of B_{ij} is not the empty set. Therefore, $\Sigma b_{ij} \rightarrow 2k\pi$, for some $k \in \mathbb{N}$, as $i \rightarrow \infty$. However, from Lemma 19 and Lemma 20, $\int_{\partial_1 U_i} w_i^* \lambda_{-\infty} - \int_{\partial_2 U_i} w_i^* \lambda_{-\infty} \leq \alpha + 1 - (\alpha - \pi) < 2\pi$. From (68) and (72) we get $2k\pi - 1 < 2\pi$, so $k = 0$. In other words, $w_i^{-1}(0) = \emptyset$. \square

Proof of compactness

In this section, we will prove w_i converges to the holomorphic building w of height 1 consisting of w_{tri} and w_{lmd} for some $w_{lmd} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)', a)$ in the sense of Symplectic Field Theory, where $a \in S^{n-1}$ is determined in Remark 6. (Refer to the appendix 6.2 for definition of convergence.) The proof uses the ideas in [5], but it is not covered by [5], because firstly, we have to deal with Lagrangian boundary condition; secondly, the almost complex structure is asymptotically cylindrical; thirdly, we need to specify to which holomorphic building the sequence converges to.

Let $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and a finite set of punctures $Z = Z_{int} \sqcup Z_{bdy}$ such that $Z_{int} = Z \cap \text{Int} D^2$ and $Z_{bdy} = Z \cap \partial D^2$.

Lemma 21. *Any finite energy punctured holomorphic map*

$$w : (D^2 - Z, \partial D^2) \rightarrow (\mathbb{R} \times S^{2n-1}, (L_1 \cup L_2) \cap \mathbb{R} \times S^{2n-1})$$

with exactly one positive puncture which corresponds to the Reeb chord γ , where γ is a Reeb chord between L_1 and L_2 with $\int_\gamma \lambda_{-\infty} = \alpha$, and given by $\gamma(t) = e^{\alpha\sqrt{-1}t}a$ for $0 \leq t \leq 1$ and some $a \in S^{n-1}$, and at least one negative puncture, has to be the trivial cylinder over γ .

Proof. Let $\overline{D^2}$ be the oriented blow up of D^2 along Z , and then the boundary of $\partial \overline{D^2}$ decomposes as $\partial \overline{D^2} = \partial_+ D^2 \cup \partial_- D^2 \cup \partial D^2$, where $\partial_+ D^2$ and $\partial_- D^2$ correspond to the positive end and negative end respectively. Stokes theorem tells us that

$$\alpha = \int_{\partial_+ D^2} w^* \lambda_{-\infty} = \int_{D^2} w^* d\lambda_{-\infty} + \int_{\partial_- D^2} w^* \lambda_{-\infty} \geq \int_{\partial_- D^2} w^* \lambda_{-\infty}.$$

However, since every Reeb orbit in $(S^{2n-1}, \lambda_{-\infty})$ has period $2k\pi$ for $k \in \mathbb{Z}^+$, and every Reeb chord has length at least α , $w(\partial_+)$ is a Reeb chord of length α . Therefore, we get $\int_{D^2} w^* d\lambda_{-\infty} = 0$, i.e. w is a trivial holomorphic cylinder over γ . \square

Lemma 22. *Any finite energy punctured holomorphic map*

$$w : (D^2 - Z, \partial D^2) \rightarrow (\mathbb{R} \times S^{2n-1}, H'_{-1})$$

with exactly one positive puncture which corresponds to the Reeb chord γ , where γ is a Reeb chord between L_1 and L_2 with $\int_\gamma \lambda_{-\infty} = \alpha$, and given by $\gamma(t) = e^{\alpha\sqrt{-1}t}a$ for $0 \leq t \leq 1$ and some $a \in S^{n-1}$, has to be a reparametrization of some $w_{lmd} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H'_{-1})', a)$.

Proof. By a similar argument as in the proof of Theorem 9, w cannot have any negative puncture. Then this follows from Theorem 8. \square

Now we have a sequence of maps w_i satisfying (67) and $E(w_i) < C < +\infty$. Let's add one auxiliary boundary marked point $\{\sqrt{-1}\}$ on the domain D^2 of each w_i to stabilize the domain, and denote the new domain as Σ_i , i.e. Σ_i is the unit disc with the marked points set $\mathfrak{M}_i = \{-1, 1, \sqrt{-1}\}$. Let's prove the compactness in this special case following the ideas in [5].

Theorem 10. *w_i converges to a holomorphic building w of height 1|1 consisting of w_{tri} and some $w_{lmd} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H'_{-1})', a)$ for some $a \in S^{n-1}$ in the sense of Symplectic Field Theory.*

Proof. The proof is sketched as follows. Let g' be a metric on $M - \{p_{12}\}$ defined as a cylindrical metric near the end of $M - \{p_{12}\}$, the standard metric g_M outside a neighborhood of p_{12} , and an interpolation between them in an annulus region on the M . We claim by repeatedly adding same number (1 or 2) of additional marked points to \mathfrak{M}_i in the following specific way, eventually at some finite step l , the marked point set becomes \mathfrak{M}_i^l , and we can achieve that $\sup_{z \in \Sigma_i - \mathfrak{M}_i^l} |\nabla w_i(z)| \cdot \rho_i^l(z)$ is uniformly bounded, where the gradient is computed with respect to g' on $M - \{p_{12}\}$ and g_i^l on $\Sigma_i - \mathfrak{M}_i^l$ which is complete hyperbolic metric of curvature -1 on $\Sigma_i - \mathfrak{M}_i^l$ making $\partial \Sigma_i$ geodesics, and corresponds to the complex structure of $\Sigma_i - \mathfrak{M}_i^l$; and $\rho_i^l(z)$ is the injective radius of the doubled Riemann Surface $\widehat{\Sigma_i - \mathfrak{M}_i^l}$ of $(\Sigma_i, \mathfrak{M}_i^l)$ along $\partial \Sigma_i$ at the point z with respect to the doubled metric \hat{g}_i^l of g_i^l (Compare Lemma 10.7 in [5]). Note that in our convention we start with step $l = 1$, i.e. $\mathfrak{M}_i^0 = \mathfrak{M}_i$.

If $\sup_{z \in \Sigma_i - \mathfrak{M}_i} |\nabla w_i(z)| \cdot \rho_i^0(z)$ is not bounded, then there exists a sequence of points $z^i \in \Sigma_i - \mathfrak{M}_i$ such that $|\nabla w_i(z^i)| \cdot \rho_i^0(z^i) \rightarrow +\infty$. Now it is easy to see that there are two cases that may happen.

Case I: There exist constants $C_1, C_2 > 0$, a subsequence of i , still denoted by i , and injective holomorphic charts $\psi_i : D^2 \rightarrow \Sigma_i - \mathfrak{M}_i$ satisfying $\psi_i(0) = z^i$ and $C_1 \rho_i^0(\psi_i(z^i)) \leq \|\nabla \psi_i(z^i)\| \leq C_2 \rho_i^0(\psi_i(z^i))$.

Case II: There exist constants $C_1, C_2 > 0$, $-1 < d_i \leq 0$ with $d_i \rightarrow 0$, and injective holomorphic charts ψ_i from $B_i := \{z \in D^2 \mid \text{Im} z \geq d_i\}$ to $\Sigma_i - \mathfrak{M}_i$, satisfying $\psi_i(0) = z^i$, $\psi_i(\{z \in D^2 \mid \text{Im} z = d_i\}) \subset \partial \Sigma_i - \mathfrak{M}_i$ and $C_1 \rho_i^0(\psi_i(z^i)) \leq \|\nabla \psi_i(z^i)\| \leq C_2 \rho_i^0(\psi_i(z^i))$.

For Case I: since $\|\nabla(w_i \circ \psi_i)\|(0) \rightarrow +\infty$, by Lemma 15, there exist $y_i \in D^2$ satisfying $y_i \rightarrow 0$ and positive constants ε_i and $R_i := \|\nabla(w_i \circ \psi_i)\|(y_i)$ satisfying $\varepsilon_i \rightarrow 0$, $R_i \rightarrow +\infty$, $\varepsilon_i R_i \rightarrow +\infty$, and $|y_i| + \varepsilon_i < 1$, such that the map $\check{w}_i(y) = w_i \circ \psi_i\left(y_i + \frac{y}{R_i}\right) : B(0, \varepsilon_i R_i) \rightarrow M \setminus \{p_{01}\}$, after an obvious translation in the \mathbb{R} direction, converges in C_{loc}^∞ to a $J_{-\infty}$ -holomorphic map $\check{w}_\infty : \mathbb{C} \rightarrow \mathbb{R} \times S^{2n-1}$ because of the conditions in formula (67). Now there are two sub-cases.

Case IA, there exist a subsequence of i , such that $\text{dist}_{g_i^0}(\psi_i(y_i), \partial\Sigma_i) > \sigma > 0$, for some constant σ . In this case, we add two marked points $\psi_i(y_i)$ and $\psi_i(y_i + \frac{1}{R_i})$ to \mathfrak{M}_i^0 and get \mathfrak{M}_i^1 . Let's denote the limit of $(\Sigma_i, \mathfrak{M}_i^k)$ by $(\Sigma^k, \mathfrak{M}^k)$. (See Appendix 6.1 for the definition of convergence.) Since $\text{dist}_{g_i^0}(\psi_i(y_i), \psi_i(y_i + \frac{1}{R_i})) \rightarrow 0$, in $(\Sigma^1, \mathfrak{M}^1)$ these two additional marked points give rise to at least an additional interior sphere bubble attached to $(\Sigma^0, \mathfrak{M}^0)$. (See Proposition 4.3 in [5] for all the possible configurations in this case.) The sphere bubble serves as the domain of \check{w}_∞ .

Case IB, $\text{dist}_{g_i^0}(\psi_i(y_i), \partial\Sigma_i) \rightarrow 0$. We add two marked points $\psi_i(y_i)$ and $\psi_i(y_i + \frac{1}{R_i})$ to \mathfrak{M}_i and get \mathfrak{M}_i^1 . In $(\Sigma^1, \mathfrak{M}^1)$ these two additional marked points give us at least an additional disc bubble together with an interior sphere bubble over that disc bubble attached to $(\Sigma^0, \mathfrak{M}^0)$. (See Figures below.) The sphere bubble serves as the domain of \check{w}_∞ .

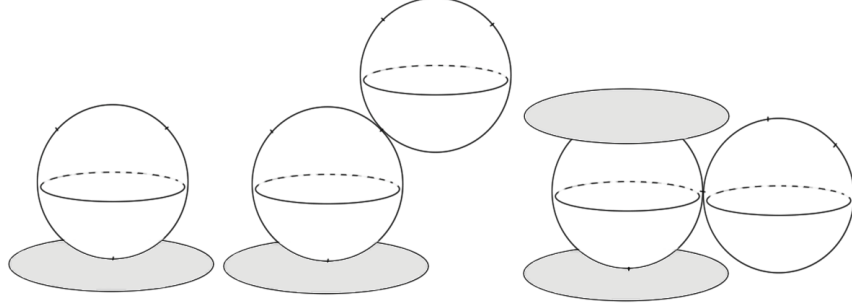
For Case II: by Lemma 1, there exist $y_i \in D^2$ satisfying $y_i \rightarrow 0$ and positive constants ε_i and $R_i := \|\nabla(w_i \circ \psi_i)\|(y_i)$ satisfying $\varepsilon_i \rightarrow 0$, $R_i \rightarrow +\infty$, $\varepsilon_i R_i \rightarrow +\infty$, $|y_i| + \varepsilon_i < 1$, and $\|\nabla(w_i \circ \psi_i)\|(y') \leq 2\|\nabla(w_i \circ \psi_i)\|(y_i)$ for all $y' \in B(y_i, \varepsilon_i)$. Now there are three sub-cases.

Case IIA, there exist a subsequence of i , still denoted by i , such that $R_i(\text{Im}y_i - d_i) \rightarrow +\infty$. This is the case similar to Case IB. We consider the map $\check{w}_i(y) = w_i \circ \psi_i\left(y_i + \frac{y}{R_i}\right) : B(0, R_i(\text{Im}y_i - d_i)) \rightarrow M \setminus \{p_{01}\}$. After an obvious translation in the \mathbb{R} direction, \check{w}_i converges in C_{loc}^∞ to a $J_{-\infty}$ -holomorphic map $\check{w}_\infty : \mathbb{C} \rightarrow \mathbb{R} \times S^{2n-1}$ because of the conditions in formula (67). We add two marked points $\psi_i(y_i)$ and $\psi_i(y_i + \frac{1}{R_i})$ to \mathfrak{M}_i^0 and get \mathfrak{M}_i^1 . In $(\Sigma^1, \mathfrak{M}^1)$ these two additional marked points give us at least an additional disc bubble together with an interior sphere bubble over that disc bubble attached to $(\Sigma^0, \mathfrak{M}^0)$. The sphere bubble serves as the domain of \check{w}_∞ .

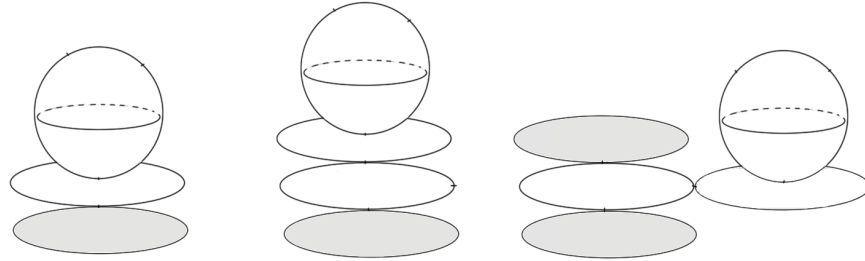
Case IIB, there exist a subsequence of i , still denoted by i , such that $R_i(\text{Im}y_i - d_i) \rightarrow c > 0$. Consider the map $\check{w}_i : B\left(0, \frac{\varepsilon_i}{2(\text{Im}y_i - d_i)}\right) \cap \mathbb{H} \rightarrow M \setminus \{p_{01}\}$ defined by $\check{w}_i(y) = w_i \circ \psi_i(\text{Re}y_i + \sqrt{-1}d_i + (\text{Im}y_i - d_i)y)$. \check{w}_i satisfies $\|\nabla\check{w}_i(\sqrt{-1})\| \rightarrow c$, $\|\nabla\check{w}_i\| \leq 2c$, and $\check{w}_i\left(B\left(0, \frac{\varepsilon_i}{2(\text{Im}y_i - d_i)}\right) \cap \partial\mathbb{H}\right) \subset L_{-\varepsilon_1, i}$. After an obvious translation in the \mathbb{R} direction, if necessary, \check{w}_i converges in C_{loc}^∞ to a $J_{-\infty}$ -holomorphic map \check{w}_∞ from \mathbb{H} to $\mathbb{R} \times S^{2n-1}$. We add one marked point $\psi_i(y_i)$ to \mathfrak{M}_i^0 and get \mathfrak{M}_i^1 . In $(\Sigma^1, \mathfrak{M}^1)$ this additional marked point give rise to at least an additional disc bubble. The disc bubble serves as the domain of \check{w}_∞ .

Case IIC, $R_i(\text{Im}y_i - d_i) \rightarrow 0$. Consider the map $\check{w}_i : B\left(0, \frac{\varepsilon_i R_i}{2}\right) \cap \mathbb{H} \rightarrow M \setminus \{p_{01}\}$ defined by $\check{w}_i(y) = w_i \circ \psi_i(\text{Re}y_i + \sqrt{-1}d_i + \frac{y}{R_i})$. Then \check{w}_i satisfies

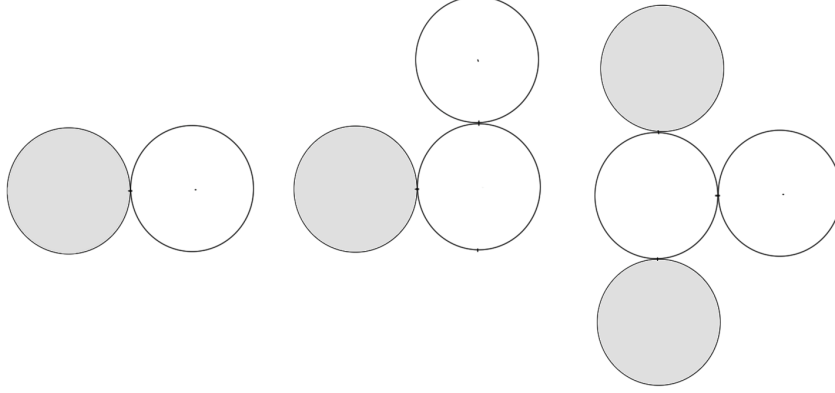
$\|\nabla \check{w}_i(\sqrt{-1})\| = 1$, $\|\nabla \check{w}_i\| \leq 2$, and $\check{w}_i(B(0, \frac{\varepsilon_i R_i}{2}) \cap \partial \mathbb{H}) \subset L_{-\varepsilon_{1,i}}$. After an obvious translation in the \mathbb{R} direction, if necessary, \check{w}_i converges in C_{loc}^∞ to a J_∞ -holomorphic map \check{w}_∞ from \mathbb{H} to $\mathbb{R} \times S^{2n-1}$. We add the additional marked points $\psi_i(\text{Re} y_i + \sqrt{-1} d_i)$ and $\psi_i(\text{Re} y_i + \sqrt{-1} d_i + \frac{1}{R_i})$ to \mathfrak{M}_i^0 and get \mathfrak{M}_i^1 . In $(\Sigma^1, \mathfrak{M}^1)$ these two additional marked points give rise to at least an additional disc bubble. The disc bubble serves as the domain of \check{w}_∞ .



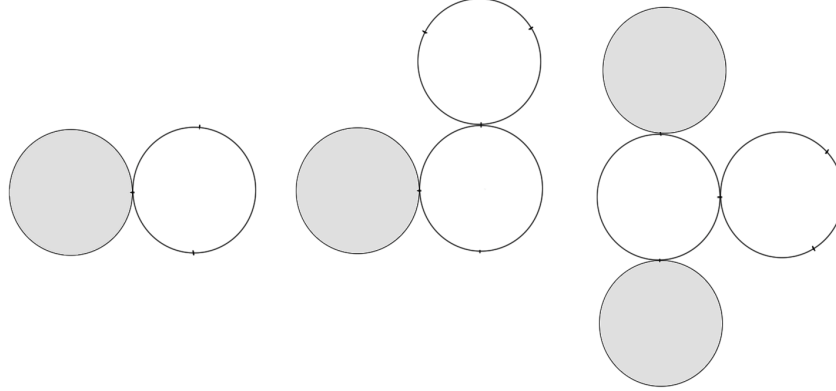
(Figure Case IA. Configurations of $(\Sigma^{k+1}, \mathfrak{M}^{k+1})$.) The shaded parts represent $(\Sigma^k, \mathfrak{M}^k)$, and the unshaded parts correspond to the additional bubbles. The first one corresponds to the situation that the two additional marked points collide to a regular interior point. The second one corresponds to the situation that the two additional marked points collide to a regular interior marked point. The third one corresponds to the situation that the two additional marked points collide to a special interior marked point.



(Figure Case IB and Case IIA. Configurations of $(\Sigma^{k+1}, \mathfrak{M}^{k+1})$.) Case IB and Case IIA have the same types of configurations. The first one corresponds to the situation that the two additional marked points collide to a regular boundary point. The second one corresponds to the situation that the two additional marked points collide to a regular boundary marked point. The third one corresponds to the situation that the two additional marked points collide to a special boundary marked point.



(Figure Case IIB. Configurations of $(\Sigma^{k+1}, \mathfrak{M}^{k+1})$.) The first one corresponds to the situation that the additional marked point collide to a regular point on the boundary. The second one corresponds to the situation when the additional marked point collide to a regular boundary marked point. The third one corresponds to the situation when the additional marked point collide to a special boundary marked point.



(Figure Case IIC. Configurations of $(\Sigma^{k+1}, \mathfrak{M}^{k+1})$.) The first one corresponds to the situation that the two additional boundary marked points collide to a regular boundary point. The second one corresponds to the situation that the two additional boundary marked points collide to a regular boundary marked point. The third one corresponds to the situation that the two additional boundary marked points collide to a special boundary marked point.

If $\sup_{z \in \Sigma_i - \mathfrak{M}_i^1} |\nabla w_i(z)| \cdot \rho(z)$ is still unbounded, then we repeat the process. It is not hard to see that different steps give rise to different bubble in the limit. By Lemma 15 and Stokes Theorem, the $\omega_{-\infty}$ -energy of \tilde{w}_∞ restricted to each domain sphere bubble is bounded away from 0. By a similar argument, it is not hard to see that the $\omega_{-\infty}$ -energy of \tilde{w}_∞ restricted to each disc bubble is also bounded away from 0. (See [14]). By Proposition 4 the process has to stop at

some finite step l .

Let $(\Sigma, \mathfrak{M}, \mathcal{D})$ be the limit of $(\Sigma_i, \mathfrak{M}_i^l)$, where \mathfrak{M} is the set of regular marked points, and \mathcal{D} is the set of special marked points, and then we get a sequence of maps $\varphi_i : \Sigma^{\mathcal{D}} \rightarrow \Sigma_i$ satisfying CRS1)-CRS3) in Appendix 6.1, where $\Sigma^{\mathcal{D}}$ is constructed from the oriented blow up of Σ along the special marked points by gluing along the special circle or half circle. We denote the set $\{z \in \Sigma_i - \mathfrak{M}_i^l | \rho_i^l(z) \geq \varepsilon\}$ by $\text{Thick}_\varepsilon(\Sigma_i - \mathfrak{M}_i^l)$ and the set $\{z \in \Sigma_i - \mathfrak{M}_i^l | \rho_i^l(z) \leq \varepsilon\}$ by $\text{Thin}_\varepsilon(\Sigma_i - \mathfrak{M}_i^l)$.

For any $\varepsilon > 0$, we have $\sup \{|\nabla w_i(z)| | z \in \text{Thick}_\varepsilon(\Sigma_i - \mathfrak{M}_i^l)\} \leq C\varepsilon$. Since $\varphi_i^* g_i^l$ converges to the hyperbolic metric g^l on $\Sigma - \mathfrak{M} \cup \mathcal{D}$ in $C_{loc}^\infty(\Sigma - \mathfrak{M} \cup \mathcal{D})$, as long as i is sufficiently large we have

$$\sup \{|\nabla(w_i \circ \varphi_i)(z)| | z \in \text{Thick}_\varepsilon(\Sigma - \mathfrak{M} \cup \mathcal{D})\} < C'\varepsilon,$$

where the norm and the injective radius are computed with respect to the hyperbolic metric g^l on $\Sigma - \mathfrak{M} \cup \mathcal{D}$. From the Gromov-Schwarz Lemma 2 we get all the higher derivatives bounds of $w_i \circ \varphi_i$ on $\text{Thick}_\varepsilon\{\Sigma - \mathfrak{M}\}$. By properly translating $w_i \circ \varphi_i$ restricted to a component of $\varphi_i^{-1}(\text{Thick}_\varepsilon(\Sigma_i - \mathfrak{M}_i^l))$, applying Ascoli-Arzelà Theorem, letting $\varepsilon \rightarrow 0$, and taking a diagonal subsequence of i , still called i , we get that up to a translation $w_i \circ \varphi_i$ converges in $C_{loc}^\infty(T)$ to $w|_T$, for any component T of $\Sigma - \mathfrak{M} \cup \mathcal{D}$.

Now let us study the convergence of $w_i \circ \varphi_i$ in $\text{Thin}_\varepsilon\{\Sigma - \mathfrak{M}\}$. We only consider the thin parts that converge to pairs of boundary special marked points, for the other cases are similar and slightly easier.

For a component T_i^ε of the ε -thin part of on $\Sigma^{\mathcal{D}}$ with respect to the hyperbolic metric $\varphi_i^* g_i^l$, assuming T_i^ε converges to a pair of boundary special marked points in \mathcal{D} as $i \rightarrow \infty$ and $\varepsilon \rightarrow 0$, there exists a conformal parametrization

$$q_i^\varepsilon : A_i^\varepsilon = [-N_i^\varepsilon, N_i^\varepsilon] \times [0, 1] \rightarrow (T_i^\varepsilon, j_i),$$

such that in the $C^\infty([0, 1])$ -sense,

$$\lim_{\varepsilon \rightarrow 0} \lim_{i \rightarrow \infty} \Theta_i \circ \varphi_i \circ q_i^\varepsilon|_{(\pm N_i^\varepsilon) \times [0, 1]} = \gamma^\pm,$$

where γ^+, γ^- are Reeb chords or constant maps, j_i is pull back complex structure via φ_i , and Θ_i is the w_i followed by the projection $\mathbb{R}^- \times S^{2n-1} \rightarrow S^{2n-1}$. We can choose the parametrization q_i^ε such that

$$|\nabla q_i^\varepsilon(x)| \leq \rho_i^l(q_i^\varepsilon(x)),$$

where the gradient is computed with respect to the flat metric in the source and hyperbolic metric $\varphi_i^* g_i^l$ in the target. This gives us the uniform gradient bound

$$\sup_{x \in A_i^\varepsilon} |\nabla(w_i \circ \varphi_i \circ q_i^\varepsilon(x))| \leq C.$$

We can choose a sequence of $\varepsilon_i \rightarrow 0$ and translate w_i by a sequence of constants a_i so that by choosing a subsequence of $w_i \circ \varphi_i \circ q_i^{\varepsilon_i} + a_i$ we get

$$\lim_{i \rightarrow \infty} (w_i \circ \varphi_i \circ q_i^{\varepsilon_i} + a_i)|_{\{\pm N_i^{\varepsilon_i}\} \times [0,1]} = \gamma^{\pm}.$$

For i large, $\Theta_i \circ \varphi_i \circ q_i^{\varepsilon_i}|_{\{\pm N_i^{\varepsilon_i}\} \times [0,1]}$ are sufficiently $C^\infty([0,1])$ close to γ^{\pm} . Since in the contact case by Stokes Theorem the ω -energy restricted to $A_i^{\varepsilon_i}$ goes to 0 as $i \rightarrow +\infty$, by Theorem 4 for every $\sigma > 0$ we get constants $c, I > 0$ so that $\Theta_i \circ \varphi_i \circ q_i(\tau, t)$ lies in the σ neighborhood of $\Theta_i \circ \varphi_i \circ q_i^{\varepsilon_i}(0, t)$ for all $(\tau, t) \in [-N_i^{\varepsilon_i} + c, N_i^{\varepsilon_i} - c] \times [0, 1]$ and $i > I$. This tells us $\gamma^+(t) = \gamma^-(t)$ for all t . This also proves that the limit map w continuously extends to the special arc.

Let $(\Sigma_{tri}, \mathfrak{M}_{tri}, \mathcal{D}_{tri})$ be the component of $(\Sigma, \mathfrak{M}, \mathcal{D})$ that corresponds to the limit of $(\Sigma_i, \mathfrak{M}_i)$. It's easy to see that from formula (67) that $w|_{\Sigma_{tri} - \mathfrak{M}_{tri} \cup \mathcal{D}_{tri}}$ equals to the J -holomorphic triangle $w_{tri} \circ \varphi_i^{-1}$. By Proposition 4, the $\omega_{-\infty}$ energy of w restricted to $\Sigma - \Sigma_{tri}$ is no greater than π . From the Stokes Theorem we know that a non-removable interior puncture will contribute $2k\pi$ to the ω energy for some $1 \leq k \in \mathbb{Z}$, a non-removable negative boundary puncture will contribute $2m\pi + \alpha$ or $(2m+1)\pi - \alpha$ to the ω energy for some $0 \leq m \in \mathbb{Z}$, and a non-removable negative boundary puncture will contribute $-2p\pi - \alpha$ or $-(2p+1)\pi + \alpha$ to the ω energy for some $0 \leq p \in \mathbb{Z}$. Thus we can easily see that the only possible configuration is that there is only one component in $\Sigma - \Sigma_{tri}$, denoted by Σ_{lmd} , and that Σ_{lmd} is a disc bubble. Thus the process of adding additional marked points stops at step $l = 2$, and it is in the Case IIB or Case IIC.

From Theorem 8 we can see that up to a reparametrization $w|_{\Sigma_{lmd} - \mathcal{D}_{lmd} \cup \mathfrak{M}_{lmd}}$ equals to a local model $w_{lmd} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_1^\alpha)', a)$. In particular, $\mathfrak{M}_{tri} = \{-1, 1, \sqrt{-1}\}$ are all removable singularity for $w|_{\Sigma_{tri} - \mathfrak{M}_{tri} \cup \mathcal{D}_{tri}}$, and near $\mathcal{D}_{tri} = \{-\sqrt{-1}\}$, $w|_{\Sigma_{tri} - \mathfrak{M}_{tri} \cup \mathcal{D}_{tri}}$ is close to a trivial cylinder over a Reeb chord γ^- of length α . \square

5.5 Gluing

To finish the proof of Theorem 7, we will construct the gluing map $glue(\epsilon_1)$:

$$\mathcal{M}((L_0, L_1, L_2), J) \times \mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)', a) \rightarrow \mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2).$$

Since $\mathcal{M}(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)', a)$ is biholomorphic to $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)', a) \cong S^{n-2}$ and we assume $w_{tri} \in \mathcal{M}((L_0, L_1, L_2), J)$ is isolated, we only need to glue w_{tri} with $w_{lmd} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)', a)$, and show that $glue(\epsilon_1)$ induces a diffeomorphism between $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{-1}^\alpha)', a)$ and $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$. In general, $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ is a fiber bundle over $\mathcal{M}((L_0, L_1, L_2), J)$ with fibers diffeomorphic to S^{n-2} . Please refer to [11] for the precise statement when w_{tri} is not isolated.

To define the $glue(\epsilon_1)$, we first preglue the two curves to get an approximately J -holomorphic strip w_{app} and then apply implicit function theorem in a

suitable setting. This is done in Pregluing in 5.5. To show $glue(\epsilon_1)$ is a diffeomorphism, the key step is to show it is surjective. This is done in Surjectivity of gluing in 5.5.

Pregluing

Let $\varsigma(\tau, t) = e^{\pi(\tau + \sqrt{-1}t)}$ be the biholomorphic map from $\mathbb{R} \times [0, 1]$ to $\mathbb{H} \subset \mathbb{C}$, and we don't distinguish between w_{lmd} and $w_{lmd} \circ \varsigma$ when there is no confusion. We also identify $\mathbb{C}^n \setminus \{0\}$ with $\mathbb{R} \times S^{2n-1}$ via the map $z \mapsto (\log |z|, z/|z|)$. And then $w_{lmd}(\tau, t) = (\varrho(\tau, t), \Theta(\tau, t))$ satisfies

$$\varrho(\tau, t) \rightarrow \alpha\tau, \quad \Theta(\tau, t) \rightarrow e^{\alpha\sqrt{-1}t}a,$$

exponentially as $\tau \rightarrow +\infty$. Denote $w_{lmd}^{\epsilon_1} := \frac{1}{2} \log |\epsilon_1| + w_{lmd}$.

Let's use $(\tau', t') \in \mathbb{R} \times [0, 1]$ as the coordinate of $\Sigma_{tri} \cong D^2 \setminus \{-\sqrt{-1}, \sqrt{-1}\}$ via $\chi(\tau', t') = \frac{\sqrt{-1}e^{\pi(\tau' + \sqrt{-1}t')} + 1}{e^{\pi(\tau' + \sqrt{-1}t')} + \sqrt{-1}}$, then $w_{tri}(\tau', t') = (\varrho'(\tau', t'), \Theta'(\tau', t'))$ satisfies

$$\varrho'(\tau', t') \rightarrow \alpha\tau' + \alpha_{tri}, \quad \Theta'(\tau', t') \rightarrow e^{\alpha\sqrt{-1}t'}a,$$

exponentially as $\tau' \rightarrow -\infty$.

Pick $R > 0$ sufficiently large so that $\alpha R - \alpha_{tri}$ is sufficiently large. Pick $\epsilon_1 < 0$ sufficiently close to 0 such that $\alpha_{tri} - \alpha R - \frac{1}{2} \log |\epsilon_1| - \log(2S_0(\alpha))$ is sufficiently large. We glue the domains $\Sigma_{lmd}^{\epsilon_1}$ and Σ_{tri} using the relation $\tau = \tau' + \alpha^{-1}(\alpha_{tri} - \frac{1}{2} \log |\epsilon_1|)$, $t = t'$ and get $\Sigma_{lmd}^{\epsilon_1} \# \Sigma_{tri}$. Let $w_{app} = w_{lmd}^{\epsilon_1} \# w_{tri}$ be the approximate solution $\Sigma_{lmd}^{\epsilon_1} \# \Sigma_{tri} \rightarrow W$ defined by:

1. if $\tau' > -R + 1$,

$$w_{app}(\tau', t') = w_{tri}(\tau', t'),$$

2. if $-R \leq \tau' \leq -R + 1$,

$$w_{app}(\tau', t') = ((1 - \beta(\tau'))(\alpha\tau' + \alpha_{tri}) + \beta(\tau')\varrho'(\tau', t'), \\ \exp_{e^{\alpha\sqrt{-1}t'}a} \left[\beta(\tau') \exp_{e^{\alpha\sqrt{-1}t'}a}^{-1} \Theta'(\tau', t') \right]),$$

3. if $-R - 1 \leq \tau' \leq -R$,

$$w_{app}(\tau', t') \\ = \left((1 - \eta(\tau'))(\alpha\tau' + \alpha_{tri}) + \eta(\tau')\varrho(\tau' + \alpha^{-1}(\alpha_{tri} - \frac{1}{2} \log |\epsilon_1|), t'), \right. \\ \left. \exp_{e^{\alpha\sqrt{-1}t'}a} \left[\eta(\tau') \exp_{e^{\alpha\sqrt{-1}t'}a}^{-1} \Theta(\tau' + \alpha^{-1}(\alpha_{tri} - \frac{1}{2} \log |\epsilon_1|), t') \right] \right),$$

4. if $\tau' \leq -R - 1$,

$$w_{app}(\tau', t') = w_{lmd}^{\epsilon_1}(\tau' + \alpha^{-1}(\alpha_{tri} - \frac{1}{2} \log |\epsilon_1|), t'),$$

where β and η are some cut-off function satisfying $\beta(-R+1) = 1$, $\beta(-R) = 0$, $|\beta'| \leq 1$, $\eta(-R) = 0$, $\eta(-R-1) = 1$, and $|\eta'| \leq 1$, and the exponential map is defined using the standard metric on S^{2n-1} .

Using the fact that J converges exponentially to the standard almost complex structure J_0 in $\mathbb{R} \times S^{2n-1}$ we can see that w_{app} satisfies

$$\begin{cases} \|\bar{\partial}_J w_{app}\| = 0 & \text{for } \tau' > -R+1 \\ \|\bar{\partial}_J w_{app}\| \leq C \left(e^{-c(\alpha R - \alpha_{tri})} + e^{-c(\alpha_{tri} - \alpha R - \frac{1}{2} \log |\epsilon_1| - \log(2S_0(\alpha)))} \right) & \text{for } \tau' \leq -R+1, \end{cases}$$

where C , c are constants independent of R , ϵ_1 and w_{lmd} , and the norm $\|\cdot\|$ is the weighted Sobolev norm computed with respect to the standard cylindrical metrics on $[0, 1] \times \mathbb{R}/\mathbb{Z}$ and on $\mathbb{R} \times S^{2n-1}$, please refer to the section 7.6 *Weighted Sobolev norm and a right inverse* in [11] for all the necessary details.

Now we can follow the general scheme of gluing and perturb w_{app} to get a regular J -holomorphic 2-gon, when R is sufficiently large and ϵ_1 is small enough. Meanwhile, we can show that $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ is regular.

Theorem 11. *For each sufficiently small ϵ_2 and ϵ_1 with $|\epsilon_1| < \epsilon_2^{100}$ we have the following*

If $\epsilon_1 < 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ is Fredholm regular and contains one element;

If $\epsilon_1 > 0$, then $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ is Fredholm regular and contains S^{n-2} parametrized family of elements.

Proof. The proof is similar to the proof in the cylindrical case. Please refer to [11]. \square

Surjectivity of gluing

In order to prove Theorem 7, we need to prove the following theorem which states that the gluing map is surjective.

Theorem 12. *Let w_i be the sequence of J -holomorphic strips satisfying (67), and then there exists a sequence of*

$$w_{lmd,i} \in \widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_{\pm 1}^\alpha)', a)$$

such that $[w_i] = [glue(\epsilon_{1,i})(w_{tri}, w_{lmd,i})]$ for all sufficient large i 's, after choosing a subsequence of i if necessary.

Proof. (Sketch) Suppose this is not true, there exists a subsequence of i , still called i , such that w_i does not come from gluing. By Theorem 10, we get a subsequence of w_i that converges to a holomorphic building of height 1|1 consisting of w_{tri} and w_{lmd} . We want to show that the holomorphic strip w_i lies in a small neighborhood of the holomorphic strip $glue(\epsilon_{1,i})(w_{tri}, w_{lmd})$. The neighborhood has to be taken under a certain strong topology so that we could apply the implicit function theorem. This can be guaranteed by the compactness

result of Theorem 10 in the Thick_ϵ -part of the domains and by Theorem 5⁶ in the Thin_ϵ -part of the domains, together with the well known results about Frenkel-Nelson coordinates description of the Deligne-Mumford compactification. For the precise norm and setting to apply the implicit function theorem, one could refer to [11]. Then we could follow the approach in [11], and define a path of maps $\Upsilon(r)$ by

$$\Upsilon(r)(s, t) = \exp_{w_i(s, t)} \left[r \cdot \exp_{w_i(s, t)}^{-1} \text{glue}(\epsilon_{1, i})(w_{tri}, w_{lmd, i}) \right].$$

$\Upsilon(r)$ satisfies $\Upsilon(0) = w_i$ and $\Upsilon(1) = \text{glue}(\epsilon_{1, i})(w_{tri}, w_{lmd})$. Since w_i lies in a small neighborhood of $\text{glue}(\epsilon_{1, i})(w_{tri}, w_{lmd})$, $\|\bar{\partial}_J \Upsilon(t)\|$ is sufficiently small. By implicit function theorem, we could perturb $\Upsilon(t)$ and get $\bar{\Upsilon}(t)$ such that $\bar{\Upsilon}(0) = \Upsilon(0)$, $\bar{\Upsilon}(1) = \Upsilon(1)$, and $\bar{\Upsilon}(t)$ is J -holomorphic.

Since $\mathcal{M}((L_{\epsilon_1}, L_0), J; w_{tri}, \epsilon_2)$ is a smooth manifold of dimension $n-2$ which equals the dimension of $\widetilde{\mathcal{M}}_0(\mathbb{H}, \mathbb{C}^n; (H_1^\alpha)', a)$, w_i comes from the gluing construction. \square

6 Appendix

The Appendix is intended to provide backgrounds for Section 4 and Section 5. We will restrict ourselves to the special case.

6.1 Bordered stable nodal Riemann surfaces

Refer to [5, 12] for details. Let $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$ be a compact possibly disconnected Riemann surface with a set \mathcal{B} of disjoint smooth circles B^i for $i = 0, 1, 2, \dots, L$ as boundaries and a set $\mathfrak{M} \sqcup \mathcal{D}$ of numbered distinct marked points. The marked points from \mathfrak{M} are regular marked point, and we allow both boundary marked points and interior marked points, i.e. $\mathfrak{M} = \mathfrak{M}_{int} \sqcup \mathfrak{M}_{bdy}$. The marked points from \mathcal{D} are special marked points, and we allow both boundary special marked points and interior special marked points, i.e. $\mathcal{D} = \mathcal{D}_{int} \sqcup \mathcal{D}_{bdy}$. The special marked points are organized in pairs: $\mathcal{D}_{int} = \{\bar{d}_1, \underline{d}_1, \dots, \bar{d}_k, \underline{d}_k\}$, $\mathcal{D}_{bdy} = \{\bar{b}_1, \underline{b}_1, \dots, \bar{b}_l, \underline{b}_l\}$. A bordered nodal Riemann surface is an equivalence class of surfaces $(S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$ under the equivalence relation: surfaces $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$ and $\mathbf{S}' = (S', j', \mathcal{B}', \mathfrak{M}', \mathcal{D}')$ are called equivalent if there exists a diffeomorphism $\varphi : S \rightarrow S'$ such that $\varphi_* j = j'$, $\varphi(\mathfrak{M}) = \mathfrak{M}'$, $\varphi(\mathcal{B}) = \mathcal{B}'$ and $\varphi(\mathcal{D}) = \mathcal{D}'$, where we assume that φ preserves the ordering of the sets \mathfrak{M} and \mathfrak{M}' , S and S' . $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$ is called decorated if for each interior special pair there is chosen an orientation reversing orthogonal map $r_i : \bar{\Gamma}_i = (T_{\bar{d}_i} S \setminus 0) / \mathbb{R}^* \rightarrow \underline{\Gamma}_i = (T_{\underline{d}_i} S \setminus 0) / \mathbb{R}^*$, we denote it by $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}, r)$. Orientation reversing orthogonal means $r(e^{\sqrt{-1}\theta} z) = e^{-\sqrt{-1}\theta} r(z)$. For $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}, r)$ we can construct the oriented blow up at each point in \mathcal{D} , identify special circles in pairs via the decoration r , identify special half circles in pairs, and get a Riemann

⁶We use the negative infinity end version of Theorem 5.

surface $\mathbf{S}^{D,r}$. We can double $\mathbf{S}^{D,r}$ along the boundary and get a smooth closed Riemann surface $\widehat{\mathbf{S}^{D,r}}$.

The signature of $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}, r)$ is the tuple

$$(g, \vec{\mu}) = (g, \mu_{int}, \mu_{bdy}^1, \mu_{bdy}^2, \dots, \mu_{bdy}^L)$$

where L is number of the boundary circles, μ_{int} is the cardinality of \mathfrak{M}_{int} , μ_{bdy}^i is the cardinality of $\mathfrak{M}_{bdy} \cap B^i$ for $1 \leq i \leq L$, and g is defined as the genus of $\widehat{\mathbf{S}^{D,r}}$. We say $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}, r)$ is connected if $\mathbf{S}^{D,r}$ is connected. A connected $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}, r)$ is called stable if

$$2g + 2\#\mathfrak{M}_{int} + \#\mathfrak{M}_{bdy} \geq 3.$$

We denote the moduli space of decorated bordered stable nodal Riemann surfaces of signature $(g, \vec{\mu})$ by $\overline{\mathcal{M}}_{g, \vec{\mu}}^s$, and the moduli space of bordered stable nodal Riemann surfaces of signature $(g, \vec{\mu})$ by $\overline{\mathcal{M}}_{g, \vec{\mu}}$. Given a bordered stable nodal Riemann surface $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$, the Uniformization Theorem asserts the existence of a unique complete hyperbolic metric $h^{j, \mathfrak{M} \cup \mathcal{D}}$ of constant curvature -1 of finite volume, in the given conformal class j on $S \setminus (\mathfrak{M} \cup \mathcal{D})$, such that B^i are geodesics, for all i .

A sequence of decorated stable nodal Riemann surfaces

$$\mathbf{S}_n = (S_n, j_n, \mathcal{B}_n, \mathfrak{M}_n, \mathcal{D}_n, r_n)$$

is said to converge to a decorated stable nodal Riemann surface

$$\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}, r)$$

if (for sufficient large n) there exists a sequence of diffeomorphisms

$$\varphi_n : S^{\mathcal{D}, r} \rightarrow S_n^{\mathcal{D}_n, r_n}$$

with $\varphi_n(\mathfrak{M}) = \mathfrak{M}_n$ and $\varphi_n(\mathcal{B}) = \mathcal{B}_n$, such that the following conditions are satisfied.

- CRS1. For every $n \geq 1$, the images $\varphi_n(\Gamma_i)$ of the special circles (half circles) $\Gamma_i \subset S^{\mathcal{D}, r}$ for $i = 1, \dots, k$, are special circles (half circles) or closed geodesics (closed geodesics after we double $S^{\mathcal{D}_n, r_n}$) with respect to the metrics $h^{j_n, \mathfrak{M}_n \cup \mathcal{D}_n}$ on $S^{\mathcal{D}_n, r_n} \setminus \mathfrak{M}_n$. Moreover, all special circles (half circles) on $S^{\mathcal{D}_n, r_n}$ are among these images.
- CRS2. $h_n \rightarrow h^{\mathbf{S}}$ in $C_{loc}^\infty(S^{\mathcal{D}, r} \setminus (\mathfrak{M} \cup \Gamma_i))$, where $h_n = \varphi_n^* h^{j_n, \mathfrak{M}_n \cup \mathcal{D}_n}$.
- CRS3. Given a component C of $\text{Thin}_\epsilon(\mathbf{S}) \subset S^{\mathcal{D}, r}$ which contains a special circle (half circles) Γ_i and given a point $c_i \in \Gamma_i$, we consider for every $n \geq 1$ the geodesic arc δ_i^n for the induced metric $h_n = \varphi_n^* h^{j_n, \mathfrak{M}_n \cup \mathcal{D}_n}$ which intersects Γ_i orthogonally at the point c_i , and whose ends are contained in the ϵ -thick part of the metric h_n . The $(C \cap \delta_i^n)$ converges as $n \rightarrow \infty$ in the C^0 -topology to a continuous geodesic for the metric $h^{\mathbf{S}}$ which passes through the point c_i .

The topology on $\overline{\mathcal{M}}_{g,\vec{\mu}}$ is the weakest topology on $\overline{\mathcal{M}}_{g,\vec{\mu}}$ for which the projection $\overline{\mathcal{M}}_{g,\vec{\mu}}^{\mathfrak{s}} \rightarrow \overline{\mathcal{M}}_{g,\vec{\mu}}$ is continuous.

Theorem 13. (*Deligne-Mumford, Wolpert*) *The spaces $\overline{\mathcal{M}}_{g,\vec{\mu}}$ and $\overline{\mathcal{M}}_{g,\vec{\mu}}^{\mathfrak{s}}$ are compact metric spaces.*

Proposition 7. (*See Proposition [5]*) *Let $\mathbf{S}_n = (S_n, j_n, \mathcal{B}_n, \mathfrak{M}_n, \mathcal{D}_n)$ be a sequence of smooth bordered stable Riemann Surfaces converging to $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$. Given a sequence of pairs of points (z_n^1, z_n^2) inside $S_n \setminus$*

6.2 Holomorphic buildings

Following the notation from 4.1, for simplicity let's assume W has only one end that is the negative end $E_- = \mathbb{R}^- \times V$. Let $L \hookrightarrow W$ be an embedded Lagrangian submanifold with respect to the symplectic form ω' . Let's define the compactification \overline{W} of W . Let $g : \mathbb{R}^- \rightarrow (-1, 0]$ be a monotone and (non-strictly) convex function which coincides with $t \mapsto e^t - 1$ for $t \in (-\infty, -1]$ and which is the identity map near 0. Define a map $G : W \rightarrow W$ by

$$G(w) = \begin{cases} (g(t), \Theta), & w = (t, \Theta) \in E_- \\ w & w \in W \setminus E_- \end{cases}$$

We define $\overline{W} =: \overline{G(W)} \subset W$.

Let $\mathbf{S} = (S, j, \mathcal{B}, \mathfrak{M}, \mathcal{D})$ be a bordered nodal Riemann surface, and $\mathfrak{M} = M \sqcup Z$. The marked points in M are called regular marked points, and the marked points in Z are called punctures consisting of boundary punctures $Z_{bdy} = Z \cap \mathcal{B}$ and interior punctures $Z_{int} = Z \cap \mathcal{B}$. We also denote the positive (negative) punctures by Z^+ (Z^-).

A holomorphic building of height 1 is defined to be a proper holomorphic map $w : (S \setminus Z, j, \mathcal{B}, \mathfrak{M}, \mathcal{D}) \rightarrow (W, J, L)$ of finite energy which sends every special pair in \mathcal{D} to one point, and sends \mathcal{B} to L . w is called stable if it satisfies: if w restricted to a component S_l of \mathbf{S} is constant, then S_l equipped with all the marked points and punctures is stable.

A holomorphic building w of height 1|1 consists of the following data:

- a holomorphic building of height 1 in W :

$$w^0 : \mathbf{S}^0 = (S^0 \setminus Z^0, j^0, \mathcal{B}^0, \mathfrak{M}^0, \mathcal{D}^0) \rightarrow (W, J, L^0),$$

- a holomorphic building of height 1 in the cylindrical manifold $\mathbb{R} \times V$:

$$w^1 : \mathbf{S}^1 = (S^1 \setminus Z^1, j^1, \mathcal{B}^1, \mathfrak{M}^1, \mathcal{D}^1) \rightarrow (\mathbb{R} \times V, J_{-\infty}, L^1),$$

- a compatible ordering of $M^0 \cup M^1$,
- we can put \mathbf{S}^0 and \mathbf{S}^1 together and pair points from Z^{0-} with points from Z^{1+} to get a bordered nodal Riemann surface $\mathbf{S} = (S^0 \sqcup S^1, j^0 \sqcup j^1, \mathcal{B}^0 \sqcup \mathcal{B}^1, M^0 \sqcup M^1 \sqcup Z^{0+} \sqcup Z^{1-}, \mathcal{D}^0 \sqcup \mathcal{D}^1 \sqcup Z^{0-} \sqcup Z^{1+})$,

- there exists a decoration r on \mathbf{S} corresponding to $Z^{0-} \sqcup Z^{1+}$ such that w^0 and the horizontal component $\Theta \circ w^1$ of w^1 fit into a continuous map $\bar{w} : \mathbf{S}^{Z^{0-} \sqcup Z^{1+}, r} \rightarrow \bar{W}$.

A holomorphic building w of height $1|1$ is called stable if both w^0 and w^1 are stable. We say w^1 is stable if not all components of \mathbf{S} are trivial cylinders and for those components where w^1 is constant map, we require those components equipped with marked points and punctures are stable.

We say a sequence of holomorphic buildings

$$w_k : \mathbf{S}_k = (S_k \setminus Z_k, j_k, \mathcal{B}_k, \mathfrak{M}_k, \mathcal{D}_k) \rightarrow (W, J, L_k)$$

of height 1 converge to a holomorphic building $w = (w^0, w^1)$ of height $1|1$, where

$$w^0 : \mathbf{S}^0 = (S^0 \setminus Z^0, j^0, \mathcal{B}^0, \mathfrak{M}^0, \mathcal{D}^0) \rightarrow (W, J, L^0)$$

$$w^1 : \mathbf{S}^1 = (S^1 \setminus Z^1, j^1, \mathcal{B}^1, \mathfrak{M}^1, \mathcal{D}^1) \rightarrow (\mathbb{R} \times V, J_{-\infty}, L^1),$$

if the following holds, there exists a sequence $M_{(k)}$ of extra sets of marked points for the curves w_k and a set M of extra marked points for the building w which have the same cardinality as $M_{(k)}$, such that $M_{(k)}$ stabilizes the underlying Riemann surface \mathbf{S}_k , M stabilizes $\mathbf{S} = \mathbf{S}^0 \sqcup \mathbf{S}^1$, and the following conditions are satisfied. Denote by $\tilde{\mathbf{S}}_k$ and $\tilde{\mathbf{S}} = \tilde{\mathbf{S}}^0 \sqcup \tilde{\mathbf{S}}^1$ the stabilized bordered nodal Riemann surfaces, and we blow up $\tilde{\mathbf{S}}$ at $Z^{0-} \sqcup Z^{1+} \sqcup \mathcal{D}^0 \sqcup \mathcal{D}^1$, glue using some decoration r , and get the surface $\tilde{\mathbf{S}}^{Z^{0-} \sqcup Z^{1+} \sqcup \mathcal{D}^0 \sqcup \mathcal{D}^1, r}$ with a conformal structure j which is degenerate along the union Γ of special circles and special half circles. Suppose there exists a sequence of diffeomorphisms $\varphi_k : \tilde{\mathbf{S}}^{Z^{0-} \sqcup Z^{1+} \sqcup \mathcal{D}^0 \sqcup \mathcal{D}^1, r} \rightarrow \tilde{\mathbf{S}}_k^{D_k, r_k}$ which satisfies the conditions CRS1-CRS3 as in 6.1, in addition, the following conditions for sufficient large k .

- CHCE1: Images $w_k \circ \varphi_k|_{\bar{\mathbf{S}}^1}$ are contained in the asymptotically cylindrical end E_- of the manifold W .
- CHCE2: There exist constants c_k , such that $\tilde{w}_k \circ \varphi_k|_{\bar{\mathbf{S}}^1}$ converges to w^1 uniformly on compact sets, where $w_k = (a_k, \Theta_k)$ and $\tilde{w}_k = (a_k + c_k, \Theta_k)$.
- CHCE3: L_k converges in the compact Hausdorff topology to L^0 ; and when restricted to E_- the translation of L_k by c_k converges to L^1 in the compact Hausdorff topology.
- CHCE4: The sequence $G \circ w_k \circ \varphi_k : \tilde{\mathbf{S}}^{Z^{0-} \sqcup Z^{1+} \sqcup \mathcal{D}^0 \sqcup \mathcal{D}^1, r} \rightarrow \bar{W}$ converges uniformly to \bar{w} .

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